ON INTEGRAL ASSEMBLY MAPS FOR LATTICES IN SL_3

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1. Statement of the result

This paper proves the integral Novikov conjecture in algebraic K-theory for lattices in the special linear group SL_3, a semisimple Lie group of rank 2. The group SL_3 has been used extensively as a trial range in extending analysis on locally symmetric spaces to "higher $\mathbb{Q}$-ranks". In a similar way, our argument uses a refinement of the methods previously successful where geometry of the group possessed some manifestation of nonpositive curvature [3, 4, 8, 9].

Theorem. *If Γ is a torsion-free lattice in SL_3 and R is an arbitrary ring, the integral assembly map $\alpha\colon h(\Gamma, K(R)) \to K(R[\Gamma])$ from the homology of the group Γ with coefficients in the K-theory spectrum $K(R)$ to the K-theory of the group ring $R[\Gamma]$ is a split injection. Here $K(A)$ stands for the nonconnective K-theory spectrum of the ring A.*

A major geometric component of the proof is the construction of a new Γ-equivariant compactification of the associated symmetric space which also contains the Borel-Serre enlargement of the symmetric space and the study of its properties.

We should point out that the topological Novikov conjecture on homotopy invariance of higher signatures has been known for torsion-free lattices in SL_3 for some time, due to various authors. It is also known, in its integral K-theoretic form as here, for cocompact lattices of SL_3, cf. [3]. On the other hand, the nonuniform lattices are not bicombable [6, 7] which excludes the possibility of applying techniques from CAT(0) geometry and its analogues to these groups. According to Margulis [18], all non-uniform lattices in SL_3 are *arithmetic*, that is, commensurable with the subgroup $SL_3(\mathbb{Z})$. The most concrete class of arithmetic groups are *congruence subgroups* defined as the kernels of surjective maps $SL_3(\mathbb{Z}) \to SL_3(\mathbb{Z}_\ell)$ induced by reduction mod ℓ for various levels ℓ. The congruence subgroups of SL_3 of all levels $\ell \neq 2$ are torsion-free, and every arithmetic subgroup contains a suitable congruence subgroup according to the solution of the congruence subgroup problem [1]. This identifies a particular cofinal system of torsion-free lattices in SL_3 to which our theorem applies.

2. Geometric preliminaries

Symmetric homogeneous spaces. It is well-known that the homogeneous quotient space $X_3 = SL_3(\mathbb{R})/SO_3(\mathbb{R})$ is a symmetric space of non-compact type.

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Let $\mathcal{P}_3$ be the space of positive definite bilinear forms

$$\mathcal{P}_3 = \{x \in GL_3(\mathbb{R}) : x = x^t,\ x > 0,\ \det(x) = 1\}.$$

Now $SL_3(\mathbb{R})$ acts on $\mathcal{P}_3$ by conjugation: $g \cdot x = gxg^t$. The isotropy group of the identity matrix $I \in \mathcal{P}_3$ is $\mathrm{stab}(I) = SO_3(\mathbb{R})$, so $\mathcal{P}_3 \cong X_3$. On the other hand, the exponential map gives an explicit diffeomorphism between the (Euclidean) space

$$\mathfrak{p} = \{x \in M_3(\mathbb{R}) : x = x^t,\ \mathrm{trace}(x) = 0\}$$

of dimension 5 and $\mathcal{P}_3$.

Recall that every non-positively curved manifold X may be compactified by attaching the ideal boundary ∂X. In the case of X_3, ∂X_3 can be identified with unit vectors in

$$T_I(X) \cong \mathfrak{p}_1 = \{Y \in M_3(\mathbb{R}) : Y = Y^t,\ \mathrm{trace}(Y) = 0,\ \mathrm{trace}(Y^2) = 1\}$$

via $x \mapsto Y(x)$, where Y is uniquely determined by $x = \gamma_Y(\infty)$ for $\gamma_Y(t) = \exp(tY)(I) = \exp(2tY)$. The point x is regular if and only if the eigenvalues of $Y(x)$ are all distinct, so the regular points form an open dense subset of ∂X_3, and the singular points form a closed nowhere dense subset.

Given a vector $Y \in \mathfrak{p}_1$, let $\lambda_1(Y) > \cdots > \lambda_k(Y)$ be the distinct eigenvalues of Y. Let $E_i(Y)$ be the eigenspace of Y associated to $\lambda_i(Y)$ and

$$V_i(Y) = \bigoplus_{j=0}^{i} E_j(Y).$$

The symmetric matrix Y (and the corresponding $x \in \partial X_3$) is completely determined by the vector $\lambda(Y) = \big(\lambda_1(Y), \ldots, \lambda_k(Y)\big) =: \lambda(x)$ and the flag $F(Y) = \big(V_1(Y), \ldots, V_k(Y)\big) =: F(x)$ in $\mathbb{R}^3$.

Theorem 2.1 (Eberlein [5]). *The action of SL_3 on ∂X_3 can be expressed by the formula*

$$g \cdot \big(\lambda(x), F(x)\big) = \big(\lambda(gx), F(gx)\big) = (\lambda(x), g \cdot F(x)),$$

where $g \cdot F(x)$ is the standard action of $g \in SL_3(\mathbb{R})$ on the flag in $\mathbb{R}^3$.

Corollary 2.2. $g \in \mathrm{stab}(x)$ *if and only if* $g \cdot F(x) = F(x)$.

The equivalence classes $W(F) = \{x : F(x) = F\} \subseteq \partial X_3$ are the *Weyl chambers* or *walls at infinity* depending on whether F is a complete flag or not. They form a tesselation of ∂X_3 which is a graph and correspond to simplices in the *Tits building* so that the boundaries of maximal 2-dimensional flats in X_3 are circular subcomplexes called *apartments* subdivided by six arcs.

Fundamental Domains. If $X = X_3$ is identified with the homogeneous space $\mathcal{P}_3$ of symmetric positive definite real 3×3 matrices (a_{ij}) up to scaling, let $[a]$ denote the class of the matrix $a = (a_{ij})$. For $\gamma \in SL_3(\mathbb{Z})$ write $[a] < [\gamma \cdot a]$ when the sequence of diagonal entries of a is smaller than the one of $\gamma \cdot a$ with respect to the lexicographic order in $\mathbb{R}^3$. This makes every orbit of $SL_3(\mathbb{Z})$ ordered, and any subset of an orbit contains a smallest point with respect to the ordering. Let Δ be the set of *reduced points* in X which are minimal in their own orbit. Minkowski showed that Δ is a fundamental domain for the action of $SL_3(\mathbb{Z})$ on X. The completion of Δ in $\bar{X}$ becomes a fundamental domain for the $SL_3(\mathbb{Z})$-action on $\bar{X}$. The domain Δ can be defined by several inequalities which can be found in [19]. Since we are interested in the dynamics of Δ at infinity, we may want to rewrite the inequalities in terms of Iwasawa coordinates which is done in [10, §2.4]. It turns out that Δ has only an *approximate box shape* at infinity. This is corrected by Grenier. His fundamental domain has *exact box shape* at infinity [10, §2.7].

Let P_0 be the standard minimal parabolic $\mathbb{Q}$-subgroup of G, let A be the maximal $\mathbb{Q}$-split torus of G contained in P_0, and K be the maximal compact subgroup in $G(\mathbb{R})$ whose Lie algebra is orthogonal (relative to the Killing form) to the Lie algebra of $A(\mathbb{R})$. Let

$$A_t = \{a \in A(\mathbb{R})^0 : \alpha(a) \leq t, \forall \alpha \in \Delta\}.$$

Recall that $P_0 = Z_G(A) \cdot R_u(P_0)$. Furthermore, $Z_G(A) \approx A \cdot F$ where F is the largest connected $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of $Z_G(A)$. From the Iwasawa decomposition, $G(\mathbb{R}) = K \cdot P(\mathbb{R})$. This yields the following decomposition:

$$G(\mathbb{R}) = K \cdot A(\mathbb{R})^0 \cdot F(\mathbb{R}) \cdot R_u P_0(\mathbb{R}).$$

Recall that a *Siegel set* in $G(\mathbb{R})$ is a set of the form

$$\Sigma_{t,\eta,\omega} = K \cdot A_t \cdot \eta \cdot \omega,$$

where η and ω are compact subsets of $F(\mathbb{R})$ and $R_u P_0(\mathbb{R})$ respectively.

Theorem 2.3 (Borel). *There are a Siegel set $\Sigma = \Sigma_{t,\eta,\omega}$ and a finite set $C \subseteq G(\mathbb{Q})$ such that $\Omega = C \cdot \Sigma$ is a fundamental set for Γ.*

Borel–Serre enlargement. In order to be able to quote from the literature, we recall a general construction of the Borel–Serre enlargement. This will be soon specialized to the case of the algebraic group SL_3.

Let G be a semisimple linear algebraic group defined over $\mathbb{Q}$ and Γ be an arithmetic subgroup of $G(\mathbb{Q})$. If Γ is torsion-free, Borel and Serre [2] enlarge the associated symmetric space X of maximal compact subgroups of $G(\mathbb{R})$ so that the action of Γ on X extends to the contractible enlargement $\bar{X}_{\mathbb{Q}}$ where it is compact. The quotient $\bar{X}_{\mathbb{Q}}/\Gamma$ is the *Borel–Serre compactification* of X/Γ.

Notation. For a linear algebraic group H defined over a subfield $k \subseteq \mathbb{C}$ we use the following notation.

$\mathcal{P}_k(H)$: parabolic k-subgroups of H,
$\mathcal{B}_k(H)$: Borel k-subgroups of H,
RH: radical of H,
S: maximal k-split torus in RH,
$A = S(\mathbb{R})^0$: Zariski connected component of the identity,
${}^0H := \bigcap_{\chi \in X^*(H)} \ker(\chi^2)$,
$\hat{L}_H = H/R_u H$: reductive Levi quotient,
π_H: canonical projection $H \to \hat{L}_H$.

Objects with "hats" are associated with $\hat{L}_H$.

$\hat{C}_H$: center of $\hat{L}_H$,
$\hat{S}_H$: maximal k-split torus in $\hat{C}_H$ with $\hat{A}_H = \hat{S}_H(\mathbb{R})^0$ as before,
$\hat{T}_H$: maximal k-split torus in $\hat{L}_H/\hat{C}_H$,
$\hat{\Delta}_H$: system of positive simple roots with respect to $\hat{T}_H$.

Let $k = \mathbb{Q}$ or $\mathbb{R}$. To each $x \in X$ is associated the Cartan involution θ_x of G and the unique θ_x-lift $\tau_x \colon \hat{L}_P(\mathbb{R}) \to P(\mathbb{R})$ which gives the θ_x-stable lifting $A_{P,x} = \tau_x(\hat{A}_P)$.

Definition 2.4. The *geodesic action* of $\hat{A}_P$ on X is given by $a \circ x = a_x \cdot x$, where $a_x = \tau_x(a) \in A_{P,x}$ is the lifting of $a \in \hat{A}_P$.

X can be viewed as the total space of a principal $\hat{A}_P$-bundle under the geodesic action. $\hat{A}_P$ can be openly embedded into $\mathbb{R}^{\hat{\Delta}-\Theta(P)}$ via

$$\hat{A}_P \longmapsto (\mathbb{R}_+^*)^{\hat{\Delta}-\Theta(P)}.$$

Let $\bar{A}_P$ be the "corner" consisting of $\hat{A}_P$ together with positive $\hat{\Delta} - \Theta(P)$-tuples where the entry ∞ is allowed with the obvious topology making it diffeomorphic to $(0,\infty]^{\hat{\Delta}-\Theta(P)}$. $\hat{A}_P$ acts on $\bar{A}_P$, and the *corner* $X(P)$ associated to P is the total space of the associated bundle $X \times_{\hat{A}_P} \bar{A}_P$ with fiber $\bar{A}_P$. Denote the common base of these two bundles by $e(P) = \hat{A}_P \backslash X$. In particular, $e(G^0) = X$. Then the *Borel–Serre enlargement*

$$\bar{X}_k = \bigsqcup_{P \in \mathcal{P}_k(G)} e(P)$$

has a natural structure of a manifold with corners in which each corner $X(P) = \bigsqcup_{Q \supseteq P} e(Q)$ is an open submanifold with corners. The action of $Q(k)$ on X extends to the enlargement $\bar{X}_k$. The faces $e(P)$, $P \in \mathcal{P}_k(G)$, are permuted under this action.

Let $q_P \colon X \to e(P)$ denote the bundle map. For any open subset $V \subseteq e(P)$ a cross-section σ of q_P over V determines a translation of V from the boundary of $\bar{X}_k$ into the interior X. For any $t \in \hat{A}_P$ put

$$\hat{A}_P(t) = \{a \in \hat{A}_P : \ a^\alpha > t^\alpha \text{ for all } \alpha \in \Delta_P\},$$

where Δ_P is the set of those simple roots with respect to a lifting of $\hat{T}_P$ that occur in $R_u P$ (transported back to $\hat{A}_P$). It is complementary to $\Theta(P)$.

Definition 2.5. For any cross-section $\sigma(V)$, a set of the form $\hat{W}(V,\sigma,t) = \hat{A}_P(t) \circ \sigma(V)$ will be called an *open set defined by geodesic influx from V into X*.

There is a natural isomorphism

$$\mu_\sigma \colon \hat{A}_P(t) \times V \xrightarrow{\cong} \hat{W}(V,\sigma,t)$$

which extends to a diffeomorphism

$$\bar{\mu}_\sigma \colon \bar{A}_P(t) \times V \xrightarrow{\cong} W(V,\sigma,t).$$

Now $W(V,\sigma,t)$ is a neighborhood of V in $\bar{X}$ with $\bar{\mu}_\sigma(\{(\infty,\dots,\infty)\} \times V) = V$. We will call it an *open neighborhood defined by geodesic influx from V into X*.

All of that done so far works for more general homogeneous H-spaces than symmetric spaces for semisimple H. Borel and Serre call them spaces of type $S-k$. For each $Q \in \mathcal{P}_k(G)$, $e(Q)$ is such a space. So

$$\overline{e(Q)}(k) = \bigsqcup_{P \in \mathcal{P}_k(Q)} e(P) = \bigsqcup_{Q \supseteq P \in \mathcal{P}_k(G)} e(P)$$

can be formed; it is diffeomorphic to the closure $\overline{e(Q)}$ of $e(Q)$ in $\bar{X}_k$. In fact, whenever $P \subseteq Q$, $\hat{A}_Q$ is canonically a subgroup of $\hat{A}_P$ so that the geodesic actions are compatible. $\hat{A}_P$ acts geodesically on $e(Q)$ through $\hat{A}_P/\hat{A}_Q$ with quotient $e(P)$. The stratum $e(P) \subseteq \overline{e(Q)}$ is the set of limit points of this geodesic action.

The parabolic $\mathbb{Q}$-subgroups index the simplices $W(P)$ of the Tits building $T(\mathbb{Q})$ of G. For $G = SL_3$ the Tits building $T_3(\mathbb{Q})$ is the simplicial complex with one vertex for each non-trivial subspace of $\mathbb{Q}^3$ and a set of vertices spanning a simplex if and only if the corresponding subspaces can be arranged into a flag. The dimensions of the strata and the corresponding Tits simplices are related via $\dim e(P) + \dim W(P) = 4$. The incidence relations among their closures reflect the structure of this building as follows:

$$e(P) \cap \overline{e(Q)} \neq \varnothing \iff e(P) \subseteq \overline{e(Q)} \iff W(Q) \subseteq W(P) \iff P \subseteq Q.$$

The minimal parabolic (Borel) $\mathbb{Q}$-subgroups correspond to the strata $e(P)$ of dimension 3, and to the maximal simplices of the building.

Remark 2.6. When B is a Borel $\mathbb{R}$-subgroup of G, we have the Iwasawa decomposition $G(\mathbb{R}) = K \cdot A_B \cdot N_B(\mathbb{R})$, where $N_B = R_u B$. Then $X \approx A_B \cdot N_B(\mathbb{R})$, and the geodesic action of A_B on X coincides with multiplication. The quotient $e(B)$ can be viewed as the underlying space of the nilpotent group $N_B(\mathbb{R})$.

Action on a Stratum. Let P be a parabolic $\mathbb{R}$-subgroup of G. The real points of the Levi quotient split as a direct product

$$\hat{L}_P(\mathbb{R}) = \hat{M}_P(\mathbb{R}) \times \hat{A}_P,$$

and there is the Langlands decomposition

$$P(\mathbb{R}) = M_{P,x} \cdot A_{P,x} \cdot L_{P,x},$$

where $M_{P,x}$ and $L_{P,x}$ are the stable lifts just like $A_{P,x}$. Recall that K_x is the stabilizer of x in $G(\mathbb{R})$ acting on X. Then $K_{P,x} = K_x \cap P(\mathbb{R})$ is the stabilizer of x in $P(\mathbb{R})$. The Borel–Serre stratum $e(P) = P(\mathbb{R})/K_{P,x} \cdot A_{P,x}$ is a space of type S for P, but it is not a symmetric space in the usual sense. Notice that it is acted upon from the left by $R_u P(\mathbb{R})$.

Definition 2.7. The quotient $\hat{e}(P)$ is the *reductive Borel–Serre stratum.*

Denote the quotient map by $\pi_P \colon e(P) \to \hat{e}(P)$. Let $\hat{K}_P = \pi_P(K_{P,x})$, then $\hat{K}_P$ is a maximal compact subgroup of $\hat{M}_P(\mathbb{R})$ and is lifted to $K_{P,x}$ by τ_x. From the Langlands decomposition,

$$\hat{e}(P) = R_u P(\mathbb{R}) \backslash P(\mathbb{R})/K_{P,x} \cdot A_{P,x} = \hat{L}_P(\mathbb{R})/\hat{K}_P \cdot \hat{A}_P \cong \hat{M}_P(\mathbb{R})/\hat{K}_P$$

is the generalized symmetric space associated to the reductive group $\hat{L}_P$.

Proposition 2.8 ([21]). *There is a diffeomorphism*

$$F \colon R_u P(\mathbb{R}) \times \hat{e}(P) \longrightarrow e(P)$$

given by

$$F(u, z\hat{K}_P\hat{A}_P) = u \cdot \tau_x(z) K_{P,x} A_{P,x} \in e(P) = P(\mathbb{R})/K_{P,x}A_{P,x}.$$

Here, $z\hat{K}_P\hat{A}_P \in \hat{e}(P) = \hat{L}_P(\mathbb{R})/\hat{K}_P\hat{A}_P$. The map F certainly depends on the choice of the basepoint x which determines the lift τ_x.

Lemma (7.8) of [11] gives a very convenient formula for the action of $P(\mathbb{R})$ on $e(P)$ in terms of the coordinates that F provides. Notice that for any $g \in P(\mathbb{R})$, $g \cdot \tau_x \mu_P(g^{-1}) \in \ker(\mu_P) = R_u P(\mathbb{R})$, so

$$g \cdot u \cdot \tau_x \mu_P(g^{-1}) = gug^{-1} \cdot g\tau_x \mu_P(g^{-1}) \in R_u P(\mathbb{R})$$

for all $g \in P(\mathbb{R})$, $u \in R_u P(\mathbb{R})$.

Lemma 2.9. *The action of* $P(\mathbb{R})$ *on* $R_u P(\mathbb{R}) \times \hat{e}(P)$ *is given by*

$$g \cdot (u, z\hat{K}_P \hat{A}_P) = (g \cdot u \cdot \tau_x \mu_P(g^{-1}), \mu_P(g) \cdot z\hat{K}_P \hat{A}_P).$$

This formula shows that $R_u P(\mathbb{R})$ acts only on the first factor by translation. Specializing it to the action of the discrete subgroup Γ_P shows that in the case of the standard (as well as any) Borel subgroup B_0, when $R_u B_0(\mathbb{R}) = N \approx e(B_0)$, the action is precisely the left multiplication action of Γ_{B_0} as a subgroup of N.

It follows from the formula that there is another equivariant enlargement where the strata are the reductive Borel–Serre strata.

Definition 2.10. The *reductive Borel–Serre enlargement* $\bar{X}_k^\rho$ ($k = \mathbb{Q}$ or $\mathbb{R}$) of X is the topological space obtained from the corresponding Borel–Serre enlargement $\bar{X}_k$ by collapsing each nilmanifold fiber of the projection $\mu_P \colon e(P) \to \hat{e}(P)$ to a point. These projections combine to give a quotient map $\mu \colon \bar{X}_k \to \bar{X}_k^\rho$.

The boundary attached to X_3 consists of hyperbolic disks stabilized by maximal parabolic subgroups and points fixed by Borel subgroups.

We will need the following explicit description of corners in two cases.

Corners for SL_2**.** The hyperbolic plane X_2 can be thought of as the open unit disk $\mathbb{E}$ in $\mathbb{C}$ or as the upper half-plane $\mathbb{H}$. Elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Q})$ act on $\mathbb{H}$ from the left as Möbius transformations $z \mapsto \frac{az+b}{cz+d}$, and the action extends to the hyperbolic boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. Recall that $\mathbb{E}$ and $\mathbb{H}$ are related via the biholomorphic Cayley mappings $\mathbb{H} \to \mathbb{E}, z \mapsto \frac{z-i}{z+i}$ and $\mathbb{E} \to \mathbb{H}, z \mapsto i\frac{1+z}{1-z}$. The *rational points* on the unit circle $\partial\mathbb{E}$ are the image of $\mathbb{Q} \subseteq \mathbb{R} \subseteq \partial\mathbb{H}$. The proper $\mathbb{Q}$-parabolic subgroups P are the stabilizers of the rational points p in $\partial\mathbb{E}$; all of them are Borel subgroups.

For each P the positive reals $\lambda \in \mathbb{R}_+$ act geodesically on X_2 by translations of magnitude $\log \lambda$ along hyperbolic geodesics in the direction of the cusp p. This is the geodesic action. The quotient map $q_P \colon X \to e(P)$ is a principal fibration with hyperbolic geodesics as fibers and the structure group $\mathbb{R}_+$. Each geodesic γ can be completed to a half-line by adding a limit point e_γ in the positive direction of the $\mathbb{R}_+$-action. Extend the action of $\mathbb{R}_+$ trivially to e_γ. The corner $X(P)$ associated to P is the total space of the associated fiber bundle with typical fiber $\gamma \cup \{e_\gamma\}$. Now $X(P) = X \cup e(P)$ where $e(P)$ is a copy of $\mathbb{R}$ associated with p which parametrizes the geodesics converging to p. Take $\bar{X}$ to be $\bigcup_P X(P)$ where P ranges over all proper $\mathbb{Q}$-parabolic subgroups.

Given a point and an open interval $y \in V \subseteq e(P)$, the restriction of a cross-section of the principal bundle q_P to V determines a neighborhood of y in $X(P)$ defined by geodesic influx from V into X which consists of all points on geodesics connecting the image of the cross-section to V (including the latter but not the former). This description makes it clear that $\bar{X}$ is a Hausdorff space. Every $g \in G$ acts as a Möbius transformation on X and sends a geodesic converging to a rational point to another hyperbolic geodesic. If $g \in \Gamma \subseteq SL_2(\mathbb{Q})$ then the new geodesic converges to a rational point and thus defines $g \cdot y \in \partial\bar{X}$.

Corners for SL_3. Choose the maximal compact subgroup $K = SO_3(\mathbb{R})$ in $G = SL_3(\mathbb{R})$, let P_0 be the Borel $\mathbb{Q}$-subgroup of G consisting of the upper triangular matrices, and let T_0 be the torus of diagonal matrices denoted by $\mathrm{diag}(t_i)$. Now $A_0 = \{\mathrm{diag}(t_i) \in T_0 : t_i > 0\}$ is the split component of T_0 which is stable with respect to the Cartan involution θ_K. Let Φ be the set of roots of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of G determined by the Cartan subalgebra $\mathfrak{a}_{\mathbb{C}}$. Since $G = SL_3$ is split over $\mathbb{Q}$, we may identify Φ and $\Phi_{\mathbb{R}}$. Choose an ordering on Φ so that the weights of $\mathfrak{a}$ are positive. The set of simple roots with respect to this ordering is $\Delta = \{\alpha_1,\ \alpha_2\}$, where α_i denotes the usual mapping t_i/t_{i+1} on T_0.

The conjugacy classes of parabolic $\mathbb{Q}$-subgroups of G are parametrized by subsets J of Δ. In particular, if Q is a maximal parabolic $\mathbb{Q}$-subgroup, then it is conjugate to a standard maximal parabolic $\mathbb{Q}$-subgroup P_j given by

$$P_j = P_{\Delta-\{\alpha_j\}} = \{(a_{ij}) \in G : a_{ik} = 0, k \le j < i\}, \quad j = 1, 2.$$

So the standard parabolic subgroups in G are either P_0 or one of

$$P_1 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in G \right\} \text{ or } P_2 = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \in G \right\}.$$

Using

$$T_{\Delta-\{\alpha_j\}} = \left(\bigcap_{\alpha_i \in \Delta, i \neq j} \ker \alpha_i \right)^0,$$

we have $P_j = Z(T_{\Delta-\{\alpha_j\}}) \cdot N_{P_j}$. If A_j is the θ_K-stable split component of P_j in the radical of P_j, and $M_j = {}^0L_j$, where $L_j = Z(A_j)$ is the Levi subgroup, then we get the Levi decomposition $P_j = M_j \cdot A_j \cdot N_j$. Explicitly, for the standard Borel subgroup P_0 we have

$$M_0 = \{\mathrm{diag}(t_i) :\ t_i = \pm 1\}, \quad N_0 = \left\{ \begin{pmatrix} 1 & n_{12} & n_{13} \\ & 1 & n_{23} \\ & & 1 \end{pmatrix} \in G \right\}.$$

For the maximal standard parabolic subgroups

$$M_1 = \left\{ \begin{pmatrix} F & \\ & \epsilon \end{pmatrix} \in G :\ F \in SL_2^{\pm}(\mathbb{R}),\ \epsilon = \pm 1 \right\},$$

where

$$SL_2^{\pm}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} :\ a, b, c, d \in \mathbb{R},\ \det = \pm 1 \right\},$$

and

$$M_2 = \left\{ \begin{pmatrix} \epsilon & \\ & F \end{pmatrix} \in G :\ F \in SL_2^{\pm}(\mathbb{R}),\ \epsilon = \pm 1 \right\}.$$

The other groups in the corresponding decompositions are:

$$A_1 = \left\{ \begin{pmatrix} a^{-1} & & \\ & a^{-1} & \\ & & a^2 \end{pmatrix} \in G :\ a \in (\mathbb{R}^*)^+ \right\}, \quad N_1 = \left\{ \begin{pmatrix} 1 & 0 & n_3 \\ & 1 & n_2 \\ & & 1 \end{pmatrix} \in G \right\},$$

$$A_2 = \left\{ \begin{pmatrix} b^2 & & \\ & b^{-1} & \\ & & b^{-1} \end{pmatrix} \in G \colon\ b \in (\mathbb{R}^*)^+ \right\},\ \ N_2 = \left\{ \begin{pmatrix} 1 & n_1 & n_3 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \in G \right\}.$$

Denote by I_3 the 3×3 identity matrix and put

$$d_1 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix},\ \ d_2 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix},$$

then, for $i = 1, 2$, M_i consists of two connected components: $M_1 = \{d_2, I_3\} \times M_1^0$ and $M_2 = \{d_1, I_3\} \times M_2^0$ with $M_i^0 \cong SL_2(\mathbb{R})$.

In this case $\operatorname{rank}_{\mathbb{Q}} G = 2$, so the Tits building is a graph whose vertices are the points and lines of the projective plane over $\mathbb{Q}$, and edges are the set of incidence relations among them. If the orbit of A_0 is identified with the first plane quadrant, then $\bar{A}_0$ is the corner with interior A_0. The orbits of A_1 and A_2 are the horizontal and vertical lines in A_0. The space $e(P_0)$ intersects each $\bar{A}_0$ at its vertex, while $e(P_i)$, $i = 1, 2$, intersect $\bar{A}_0$ in the vertical and horizontal lines of its boundary. Fixing a vertex $W(P)$ let C range over all chambers having $W(P)$ as an endpoint, i.e., over all $W(B)$ for Borel $\mathbb{Q}$-subgroups $B < P$. These are indexed by the projective line over $\mathbb{Q}$. The corresponding 3-dimensional strata $e(B)$ are disjoint in the boundary of the 4-dimensional space $\overline{e(P)}$. Everywhere in this paragraph the field $\mathbb{Q}$ may be replaced by $\mathbb{R}$. The resulting combinatorial object $T_3(\mathbb{R})$ is dual to the combinatorics of the maximal Satake compactification of X rather than the noncompact rational enlargements of Borel–Serre and Satake.

Topological Properties. Recall that there is a continuous map $\mu \colon \bar{X}_{\mathbb{R}} \to \bar{X}^{\rho}_{\mathbb{R}}$ where the reductive Borel–Serre enlargement $\bar{X}^{\rho}_{\mathbb{R}}$ coincides with the maximal Satake compactification X^S of X, cf. [22], Remark 7.12. For $G = SL_3$ the map μ is the identity on X_3 and projects each $e(gP_ig^{-1}) \cong \hat{e}(gP_ig^{-1}) \times R_u(gP_ig^{-1})(\mathbb{R})$, $i = 1, 2$, onto the first factor. Using the comparison with the maximal Satake compactification, the space $\bar{X}^{\rho}_{\mathbb{R}}$ is certainly compact and Hausdorff. We will also need to use its homological triviality.

Theorem 2.11. *The space X^S is Chogoshvili-acyclic.*

Proof. Recall that the Chogoshvili homology theory is the unique extension of the Steenrod–Sitnikov homology to compact Hausdorff spaces from the category of compacta satisfying the three axioms of Berikashvili (see [8, §4.3, §11.3]). So X^S needs to be Steenrod-acyclic. We denote the Steenrod–Sitnikov homology by $H_*(\_)$. We will use the following version of the Vietoris–Begle theorem.

Lemma 2.12 (Nguen Le Ahn [16]). *Let $f \colon X \to Y$ be a continuous surjective map of metrizable compacta so that*

$$\widetilde{H}_i(f^{-1}(y); G) = 0$$

for all $y \in Y$, $i \leq n$. Then if G is a countable group, the induced homomorphism

$$H_q(f) \colon H_q(X; G) \longrightarrow H_q(Y; G)$$

is an isomorphism for $0 \leq q \leq n$ and an epimorphism for $q = n + 1$.

According to [12, Theorem 1], X^S is homeomorphic to the Martin compactification $X^M(\lambda_0)$ of X at the bottom of the positive spectrum λ_0. There is also the Karpelevič compactification X^K [14] which is defined inductively and maps equivariantly onto $X^M(\lambda_0)$ (see [12, Theorem 4]). Lemma 2.12 applies to this map $f\colon X^K \to X^M(\lambda_0)$ because the fibers of f are easily seen to be genuinely contractible using the result of Kushner [15] that X^K is homeomorphic to a ball. The same result applied to X^K itself shows that all of the spaces in $D^n \cong X^K \to X^K(\lambda_0) \cong X^S$ are Chogoshvili-acyclic. $\triangledown$

3. Compactification of X_3

This new compactification of the symmetric space X_3 is different from the classical compactifications in that it also contains the Borel–Serre enlargement $E\Gamma = \bar{X}_{\mathbb{Q}} \subseteq \bar{X}_{\mathbb{R}}$ as an open dense subspace.

The nature of the construction is inductive. The truly basic case is that of the one-dimensional Lie group $L = \mathbb{R}$ acting on itself by addition. The associated symmetric space is L, and the equivariant compactification we want is the obvious completion $\hat{L} = L \cup \{-\infty, +\infty\}$.

Next, we deal with corners of hyperbolic disks, as in section 2. Consider the standard parabolic $\mathbb{R}$-subgroup P_0 of $G = SL_2$. The corresponding reductive stratum in $\bar{X}^{\rho}_{\mathbb{R}}$ is a point, and

$$e(P_0) \cong R_u P_0(\mathbb{R}) = \left\{ \begin{pmatrix} 0 & x \\ & 0 \end{pmatrix} : \; x \in \mathbb{R} \right\} \cong L.$$

A proper parabolic subgroup $P \subseteq SL_2$ stabilizes a point $p(P) \in \partial\mathbb{E}$ and permutes the geodesics abutting to $p(P)$. The stratum $e(P) \cong L$ parametrizes those geodesics. Complete each stratum as $\varepsilon(P) \cong \hat{L}$. The resulting set is $\hat{X}$ in which every corner $X(P)$ is declared to be open. So typical open neighborhoods of $z \in e(P)$ in $\hat{X}$ are the open neighborhoods of z in $X(P)$. Given a line $e(P)$ and one of its endpoints y, a typical open neighborhood of y consists of

- y itself and an open ray in $e(P)$ asymptotic to y,
- an open (Euclidean) set U in $\mathbb{E}$ bounded by the hyperbolic geodesic γ abutting to $p(P)$ representing the origin of the ray in $e(P)$—the one which is the union of geodesics representing other points of the ray,
- points in various $e(B)$, $B \in \mathcal{P}_{\mathbb{R}}$, such that $p(B)$ is on the arc in $\partial\mathbb{E}$ connecting $p(P)$ with $p(R)$, the opposite end of γ, which are represented by geodesics with a subray inside U,
- each endpoint of the corresponding $\varepsilon(B)$ if $B \neq P$, R, and
- the endpoint of $\varepsilon(R)$ which is the limit of a ray in $e(R)$ contained in the set from (3).

This choice generates a well-defined topology consisting of subsets which contain a neighborhood of each of its members. For example, the intersection of finitely many rays $\{\rho_i\} \subseteq \varepsilon(P)$ converging to the same end y of $\varepsilon(P)$ is the smallest ray, and the corresponding neighborhood of y is the intersection of the neighborhoods determined by $\{\rho_i\}$. In other words, the totality of all neighborhoods constructed above forms a base.

With the topology on $\hat{X}$ generated as above, the subspace $X \subseteq \hat{X}$ has the hyperbolic metric topology, and $\delta X = \hat{X} - X$ is simply $S^1 \times I$ with an analogue of the lexicographic order topology, see Example 6.2.7 in [8] for a complete description. An interesting property is that δX is compact but not separable and, therefore, not metrizable.

Minimal Borel–Serre strata. For a torsion-free arithmetic subgroup Γ of $G(\mathbb{Q})$ and any rational Borel subgroup B, $\Gamma \cap B(\mathbb{Q})$ is the largest subgroup which acts in $e(B)$. It is a cocompact nilpotent discrete subgroup $\Gamma \cap N_B$ of the nilpotent component N_B in the Langlands decomposition of $B(\mathbb{R})$. Indeed, for the standard Borel subgroup P_0 of $G = SL_3$ and $N_0 = N_{P_0}$, $\Gamma \cap N_0$ is the discrete Heisenberg group, and N_0/Γ_0 is the compact 3-dimensional Heisenberg nilmanifold.

As in the construction of $\hat{X}(SL_2)$, we first compactify each $e(B)$, $B \in \mathcal{B}_{\mathbb{R}}$, $\Gamma_B = \Gamma \cap B(\mathbb{R})$-equivariantly, then provide the new points with neighborhoods which form a part of the basis for the topology on $\hat{X}$. In fact, it suffices to compactify $e(P_0)$ and extend the construction equivariantly to other strata.

Recall that $e(P_0)$ can be identified with N_0 (Remark 2.6) and that Γ_0 acts on N_0 via left multiplication (Lemma 2.9). The discrete Heisenberg group Γ_0 has the well-known presentation

$$\Pi = \langle a, b, c \mid aca^{-1}c^{-1},\ bcb^{-1}c^{-1},\ caba^{-1}b^{-1} \rangle.$$

This is the simplest example of a discrete nilpotent but non-abelian group. Each element of Π can be written uniquely in the form $a^m b^k c^l$, and the identification of Π with the subgroup $\Gamma_0 \subseteq N_0$ is given by the mapping

$$a^m b^k c^l \longmapsto \begin{pmatrix} 1 & m & l \\ & 1 & k \\ & & 1 \end{pmatrix}.$$

Γ_0 contains two subgroups

$$M = \left\{ \begin{pmatrix} 1 & m & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\} \cong \mathbb{Z} \quad \text{and} \quad L = \left\{ \begin{pmatrix} 1 & 0 & l \\ & 1 & k \\ & & 1 \end{pmatrix} \right\} \cong \mathbb{Z}^2.$$

M acts on L by

$$m \cdot z = mzm^{-1} = (k, l + mk)$$

for $m \in M$ and $z = (k, l) \in L$ making Γ_0 into the semidirect product of M and L with multiplication

$$(m, z) \cdot (m', z') = (m + m', z + m \cdot z').$$

Systematizing the convenient notation, let us denote the matrix

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \in N_0$$

by the ordered triple (x, y, z). In this notation the action of Γ_0 on N_0 is given by

$$(m, k, l) \cdot (x, y, z) = (m + x, k + y, l + z + my).$$

A fundamental domain for this action is

$$\mathcal{D} = \{(x, y, z) \in N_0 : 0 \le x, y, z \le 1\}.$$

Definition 3.1. Let $\epsilon(P_0) = N_0 \cup \partial N_0$ be the ideal compactification of $e(P_0) = N_0$ viewed as the Euclidean space $\{(a, b, c)\} = \mathbb{R}^3$ with the standard flat metric.

A straight line in the flat space N_0 can be expressed in terms of our coordinates as $(x_1 + x_2t, y_1 + y_2t, z_1 + z_2t)$ with the parameter $t \in \mathbb{R}$. Now Γ_0 acts on the set of such lines:

$$(m,k,l) \cdot (x_1 + x_2t, y_1 + y_2t, z_1 + z_2t) =$$
$$(m + x_1 + x_2t, k + y_1 + y_2t, l + my_1 + z_1 + (my_2 + z_2)t).$$

From this equation we see that the parallelism class of lines with $x_2 = y_2 = 0$ is invariant under Γ_0. The same is true for the class of lines with $y_2 = z_2 = 0$. Observe that this left action extends to the ideal boundary of N_0—the points in ∂N_0 corresponding to the two opposite directions of the line $(0,0,t)$ are fixed by Γ_0, and the open meridian semicircles connecting the two points are orbits. The action is continuous since the coefficients at t are degree one polynomials in the coordinates of the directional vector (x_2, y_2, z_2).

The right action of Γ_0 on lines is given by

$$(x_1 + x_2t, y_1 + y_2t, z_1 + z_2t) \cdot (m,k,l) =$$
$$(m + x_1 + x_2t, k + y_1 + y_2t, l + kx_1 + z_1 + (kx_2 + z_2)t).$$

Notice that the same poles with $x_2 = y_2 = 0$ get fixed. Also, the sets of equivalence classes with either $x_2 = 0$ or $y_2 = 0$ are invariant under the right action. In fact, all right Γ_0-orbits in the former set are points.

Definition 3.2 (Cubical Cellular Decompositions). Let $I^3 = [-1,1]^3$ be the 3-dimensional cube embedded in $\mathbb{R}^3$. It has eight vertices indexed by various triples with entries either 1 or -1. Let us denote this set by $V_{(-1)}$. We also say that $V_{(-1)}$ is derived from $I_{(-1)} = \{\pm 1\}$ and write this as $V_{(-1)} = I^3_{(-1)}$. Now define the following subsets of I:

$$I_{(0)} = \{-1, 0, 1\}, \quad I_{(1)} = \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}, \quad \ldots$$

where

$$I_{(i)} = \left\{-1, \ldots, \frac{k}{2^i}, \frac{k+1}{2^i}, \ldots, 1\right\}, \quad k \in \mathbb{Z}, \quad -2^i \le k \le 2^i,$$

for $i \in \mathbb{N}$. We also get the corresponding derived subsets of I^3:

$$V_{(0)}, V_{(1)}, \ldots, V_{(i)} = \{v_i(s_1, s_2, s_3)\} = I^3_{(i)}, \ldots$$

where

$$v_i(s_1, s_2, s_3) \overset{\text{def}}{=} \left(\frac{s_1}{2^i}, \frac{s_2}{2^i}, \frac{s_3}{2^i}\right), \quad s_j \in \mathbb{Z}, \quad -2^i \le s_j \le 2^i.$$

At each stage $V_{(i)}$ is the set of vertices of the obvious cellular decomposition of I^3, where the top dimensional cells are 3-dimensional cubes with the j-th coordinate projection being an interval

$$\left[\frac{k_j}{2^i}, \frac{k_j + 1}{2^i}\right] \subseteq I, \quad 1 \le j \le i.$$

These cells can be indexed by the triples $\{(k_1, k_2, k_3) : -2^i \le k_j < 2^i\}$, the coordinates of the lexicographically smallest vertex, $2^{3(i+1)}$ triples at all.

These decompositions behave well with respect to the collapse performed in the boundary of the cube I^3 which contracts faces

$$\{(x_1, *, *) \in I^3 : x_1 = \pm 1\} \longrightarrow (\pm 1, \flat, \flat).$$

The result is a topological ball B^3 with the CW-structure consisting of two cells of dimension 0, four cells of each dimension 1 and 2, one 3-dimensional cell and a continuous collapse $\rho\colon I^3 \to B^3$. Notice that every old derived cubical CW-structure in I^3 induces a CW-decomposition of the image in the obvious way.

The right action of N_0 or Γ_0 on ∂N_0 fixes the poles with $x_2 = y_2 = 0$, and the circles of points with $x_2 = 0$ or $y_2 = 0$ are themselves invariant sets. In particular, the four open arcs complementary to the fixed poles are invariant, and the four connected components of the complement to the circles are also invariant. Consider the (-1)-st derived decomposition of I^3 and the corresponding CW-structure in B^3. The cells in ∂B^3 are in bijective correspondence with the invariant cells just described. We will refer to this isomorphism of CW-structures as $\Upsilon\colon \partial B^3 \to \partial N_0$.

There are cubical analogues of links and stars of the usual simplicial notions. Thus the *star* of a vertex is the union of all cells which contain the vertex in the boundary. The *open star* is the interior of the star. For the i-th derived cubical decomposition, the open star of the vertex $v_i(s_1, s_2, s_3)$ will be denoted by $\mathrm{Star}^{\mathrm{o}}(v_i(s_1, s_2, s_3))$. These sets form the *open star covering* of I^3.

By *vertices* in ∂N_0 we mean the image $\Upsilon\rho(V_{(n)} \cap \partial I^3)$. Let $v \in \Upsilon\rho(V_{(n)} \cap \partial I^3)$ then

$$\mathrm{Star}^{\mathrm{o}}((\Upsilon\rho)^{-1}(v) \cap V_{(n)}) = \bigcup v_n \in V_{(n)}\Upsilon\rho(v_n) = v\,\mathrm{Star}^{\mathrm{o}}(v_n)$$

is an open neighborhood (the open star) of $(\Upsilon\rho)^{-1}(v)$, and, in fact,

$$\mathrm{Star}^{\mathrm{o}}_n(v) \overset{\text{def}}{=} \Upsilon\rho(\mathrm{Star}^{\mathrm{o}}(\rho^{-1}\Upsilon^{-1}(v) \cap V_{(n)}))$$

is an open neighborhood of v which we call the *open star* of v. The map $\Upsilon\rho$ is bijective in the interior of I^3, so $\mathrm{Star}^{\mathrm{o}}_n(v)$ can be defined by the same formula for $v \in \Upsilon\rho(V_{(n)} \cap \mathrm{int} I^3)$.

Maximal Borel–Serre Strata. We proceed to Γ_{P_i}-equivariantly enlarge each of $e(P_i)$, $i = 1, 2$. Recall the projection map

$$\mu_{P_i}\colon e(P_i) = R_u P_i(\mathbb{R}) \times \hat{e}(P_i) \longrightarrow \hat{e}(P_i).$$

Proposition 3.3. *For each $P \in \mathcal{P}_{\mathbb{R}}(G)$, the principal $R_uP(\mathbb{R})$-fibration μ_P extends to a principal fibration*

$$\bar{\mu}_P\colon \overline{e(P)} \longrightarrow \overline{\hat{e}(P)}.$$

Proof. This can be seen from [2, §§2.8, 3.10, 5.2, 7.2(iii)]. Let $Q \subseteq P$ be proper parabolic subgroups with unipotent radicals $R_uQ \supseteq R_uP$, then Q determines a parabolic subgroup

$$Q^P = \pi_P(Q) = Q/R_uP \subseteq \hat{L}_P = P/R_uP$$

with unipotent radical $R_uQ^P = R_uQ/R_uP$. Now A_{Q^P} is canonically identified with $A_{P,B}$ (in the notation of Borel and Serre), and the geodesic actions of A_Q on $e(P)$ and $\hat{e}(P)$ commute with μ_P. So $X_P(Q)$ is a principal $R_uP(\mathbb{R})$-bundle over $X_{\hat{L}_P}(Q^P)$, and the projection $\tau_Q\colon X_P(Q) \to X_{\hat{L}_P}(Q^P)$ extends μ_P. These fibrations τ_* are compatible with the order in the lattice $\mathcal{P}(P)$ in the sense that for each pair $Q_1 \subseteq Q_2 \subseteq P$ the restriction of τ_{Q_1} to $e(Q_2)$ is the projection of a principal $R_uP(\mathbb{R})$-fibration with base $e(Q_2^P)$. So the principal fibrations τ_* are also compatible with the inclusions $X(Q_2) \hookrightarrow X(Q_1)$ and match up to give a principal fibration for $\overline{e(P)}$ over $\overline{\hat{e}(P)}$. $\triangledown$

The real version

$$\bar{\mu}_{P,\mathbb{R}} \colon \overline{e(P)}_{\mathbb{R}} \longrightarrow \overline{\hat{e}(P)}_{\mathbb{R}}$$

of the principal fibration $\bar{\mu}_P$ has the related product structure which extends $\bar{\mu}_P$. The total space $\overline{e(P_i)}_{\mathbb{R}}$ is the first partial enlargement of $e(P_i)$.

The other enlargement is obtained by compactifying each flat fiber of the principal fibration μ_{P_i} with its ideal boundary. In other words, we embed $e(P_i)$ in

$$\varepsilon(P_i) \overset{\text{def}}{=} \varepsilon\big(R_u P_i(\mathbb{R})\big) \times \hat{e}(P_i).$$

The formula from Lemma 2.9 shows that $P_i(\mathbb{R})$ acts on $e(P_i)$ by bundle automorphisms. It can be used to see that in general the action extends to $\varepsilon(P_i)$. In our low-dimensional situation, the following concrete calculation may be more satisfying.

Proposition 3.4. *The inclusion*

$$e(P_1) \subseteq \varepsilon\big(R_u P_1(\mathbb{R})\big) \times \hat{e}(P_1)$$

is a $P_1(\mathbb{R})$-equivariant enlargement.

Proof. The added points are the parallelism classes of rays in fibers of μ_{P_i}. We are going to check that the bundle automorphisms from $P_1(\mathbb{R})$ map lines to lines in a way which preserves the parallelism relation and that the formulae for the action are polynomial.

A line in the μ_{P_i}-fiber over $\hat{z} = z\hat{K}_{P_1}\hat{A}_{P_1}$ can be parametrized as

$$(\{u_t\}, \hat{z}) = \big(\{0, y_1 + y_2 t, z_1 + z_2 t)\}, \hat{z}\big) \subseteq R_u P_1(\mathbb{R}) \times \{\hat{z}\}.$$

An element $g \in P_1(\mathbb{R})$ maps the fiber to $R_u P_1(\mathbb{R}) \times \{\mu_{P_1}(g) \cdot \hat{z}\}$. If $g = (g_{ij})$, $1 \le i, j \le 3$, $g_{31} = g_{32} = 0$, then

$$g \cdot (0, y_1 + y_2 t, z_1 + z_2 t) \cdot g^{-1} =$$
$$(0, g_{11}z_1 + g_{12}y_1 + (g_{11}z_2 + g_{12}y_2)t, g_{21}z_1 + g_{22}y_1 + (g_{21}z_2 + g_{22}y_2)t).$$

The $R_u P_1(\mathbb{R})$-coordinate of $g \cdot (u_t, \hat{z})$ is given by $g u_t g^{-1} \cdot g\tau_x \mu_{P_1}(g^{-1})$, where $g\tau_x\mu_{P_1}(g^{-1}) \in R_u P_1(\mathbb{R})$. Since the right multiplication in $R_u P_1(\mathbb{R})$ is simply the coordinatewise addition, $g \cdot (u_t, \hat{z})$ is a line with the slope coefficients depending polynomially only on y_2 and z_2. $\triangledown$

Proposition 3.5. *The inclusion*

$$e(P_2) \subseteq \varepsilon\big(R_u P_2(\mathbb{R})\big) \times \hat{e}(P_2)$$

is a $P_2(\mathbb{R})$-equivariant enlargement.

Proof. The proof is identical to that of Proposition 3.4 but the formulae are different. This reflects the non-symmetric equivariance in the corner associated to P_0. If $g = (g_{ij})$, $1 \le i, j \le 3$, $g_{21} = g_{31} = 0$, and

$$u_t = (x_1 + x_2 t, 0, z_1 + z_2 t) \in R_u P_2(\mathbb{R}),$$

then

$$g u_t g^{-1} = g_{11}\left(\frac{g_{33}}{D}(x_1 + x_2 t) - \frac{g_{32}}{D}(z_1 + z_2 t), 0, \frac{g_{22}}{D}(z_1 + z_2 t) - \frac{g_{23}}{D}(x_1 + x_2 t)\right),$$

where $D = g_{22}g_{33} - g_{23}g_{32}$. $\triangledown$

Notation. We will treat the two maximal standard proper parabolic subgroups P_1 and P_2 simultaneously using notation P for either of them. Correspondingly, the notation that appeared heretofore with the subscripts 1 and 2 is used without a subscript.

The closures of proper Borel–Serre strata for SL_2 are disjoint in the boundary of $\hat{X}_{\mathbb{R}}(SL_2)$, and so are the closures of the strata in $\overline{e(P)_{\mathbb{R}}} - e(P)$, where $P(\mathbb{R})$ operates by automorphisms. It also acts on the disjoint union of the ideal compactifications $\varepsilon(B)$, i.e., on

$$\delta(P) \stackrel{\text{def}}{=} \varepsilon(P_0) \times_{P_0(\mathbb{R})} P(\mathbb{R}).$$

Definition 3.6. Define the set $e(P)^{\wedge} = \varepsilon(P) \sqcup \delta(P)$.

The topology in $e(P)^{\wedge}$ can be introduced by specifying the neightborhoods of each individual point.

Definition 3.7. The space $\varepsilon(P)$ has the product topology. For $y \in \varepsilon(P)$ let

$$\mathcal{N}(y) = \{\mathcal{O} \subseteq e(P)^{\wedge} : \mathcal{O} \text{ contains an open neighborhood of } y \text{ in } \varepsilon(P)\}.$$

Also, $\overline{e(P)_{\mathbb{R}}} \subseteq e(P)^{\wedge}$ is the Borel–Serre construction over $\mathbb{R}$ and has topology in which each corner $X(B)$, $B \in \mathcal{P}_{\mathbb{R}}(P)$, is open. For $y \in \overline{e(P)_{\mathbb{R}}}$ let

$$\mathcal{N}(y) = \{\mathcal{O} \subseteq e(P)^{\wedge} : \mathcal{O} \text{ contains an open neighborhood of } y \text{ in } \overline{e(P)_{\mathbb{R}}}\}.$$

Notation. Given an open subset $U \subseteq \varepsilon(B)$, let $\mathcal{O}(U) = q_{P,B}^{-1}(V)$, the total space of the restriction to $V = U \cap e(B)$ of the trivial bundle $q_{P,B}$ over $e(B)$ with fiber $A_{P,B}$. If U is any open subset of $\delta(P)$, let

$$\mathcal{O}(U) = \bigcup_{B \in \mathcal{P}_{\mathbb{R}}} \mathcal{O}(U \cap e(B)).$$

Let $y \in \partial e(B)$. Recall the map $p \colon e(P)^{\wedge} \to \varepsilon X(M^0)$. If U is any open subset of $\delta(P)$, define

$$\begin{aligned} C(U) = &\{z \in \varepsilon(P) \cup \overline{e(P)_{\mathbb{R}}} : \exists\, \mathcal{O} \in \mathcal{N}(z) \text{ such that } \mathcal{O} \cap e(P) \subseteq \mathcal{O}(U)\} \cup \\ &\{z \in \delta(P) \backslash \overline{e(P)_{\mathbb{R}}} : \exists \text{ open } U' \subseteq \delta(P) \text{ such that } z \in U' \text{ and } \mathcal{O}(U') \subseteq \mathcal{O}(U)\} \end{aligned}$$

and let

$$\begin{aligned} \mathcal{N}(y) = \{\mathcal{O} \subseteq e(P)^{\wedge} : &\exists \text{ open set } U \subseteq \delta(P) \text{ with } y \in U \text{ and} \\ &\exists \text{ open set } V \subseteq \varepsilon X(M^0) \text{ with } C(U) \cap p^{-1}V \subseteq \mathcal{O}\}. \end{aligned}$$

This defines a system of neighborhoods $\mathcal{N}(y)$ for every $y \in e(P)^{\wedge}$.

For a subset $S \subseteq e(P)^{\wedge}$, let $\mathcal{N}(S) = \{\mathcal{O} \subseteq e(P)^{\wedge} : \mathcal{O} \in \mathcal{N}(y)$ for every $y \in S\}$. We call S *open* if $S \in \mathcal{N}(S)$. It is routine to check that the open subsets of $e(P)^{\wedge}$ form a topology. It is also easy to see that $\hat{X}(P)$ is compact Hausdorff and Čech-acyclic as in section 2.

Definition of $\hat{X}(SL_3)$. The conjugation action of $SL_3(\mathbb{R})$ permutes the Borel–Serre strata associated to the three standard proper parabolic subgroups P_i, $i = 0, 1, 2$. Each stratum may be compactified as in Definitions 3.1 and 3.6. Denoting $\varepsilon(P_0)$ by $e(P_0)^{\wedge}$, define

$$Y_i = e(P_i)^{\wedge} \times_{P_i(\mathbb{R})} SL_3(\mathbb{R}).$$

Since $e(P_0)\hat{}\subseteq e(P_1)\hat{}$, $e(P_0)\hat{}\subseteq e(P_2)\hat{}$, we have $Y_0\subseteq Y_1$, $Y_0\subseteq Y_2$ and can form

$$\delta(X_3)\overset{\text{def}}{=}Y_1\cup_{Y_0}Y_2.$$

Definition 3.8. $\hat{X}=\bar{X}_{\mathbb{R}}\cup\delta X=X\sqcup\delta X$.

The space $\bar{X}_{\mathbb{R}}$ is the Borel–Serre construct and has the topology in which each corner $X(P)$ is open. For $y\in\bar{X}_{\mathbb{R}}$ let

$$\mathcal{N}(y)=\{\mathcal{O}\subseteq\hat{X}:\mathcal{O}\text{ contains an open neighborhood of }y\text{ in }\bar{X}_{\mathbb{R}}\}.$$

Given a maximal parabolic $\mathbb{R}$-subgroup P and an open subset $U\subseteq e(P)\hat{}$, let $\mathcal{O}(U)=q_P^{-1}(V)$, the total space of the restriction to $V=U\cap e(P)$ of the trivial bundle q_P over $e(P)$ with fiber $\hat{A}_P$. If U is any open subset of δX, let

$$\mathcal{O}(U)=\bigcup_{B\in\mathcal{P}_{\mathbb{R}}}\mathcal{O}\big(U\cap e(P)\big).$$

Notation. Let $y\in\partial e(B)$ for some $B\in\mathcal{B}_{\mathbb{R}}$ and let P, $Q\in\mathcal{P}_{\mathbb{R}}$ such that $B\subseteq P$, Q. Then for any open neighborhood Ω of y in δX, $\Omega\cap\varepsilon(B)$ contains an open neighborhood U of y in $\varepsilon(B)$ such that $q_P^{-1}(V)\cup q_Q^{-1}(V)\subseteq\Omega$, where $V=U\cap e(B)$, and $q_P\colon e(P)\to e(B)$ and $q_Q\colon e(Q)\to e(B)$ are the associated bundles with fibers $\hat{A}(P,B)$ and $\hat{A}(Q,B)$ respectively. The point is that

$$\mathcal{O}\big(V\cup q_P^{-1}(V)\cup q_Q^{-1}(V)\big)=q_B^{-1}(V).$$

It is convenient to denote this set also by $\mathcal{O}(U)$ even though $U\subseteq\varepsilon(B)$ is not open in δX.

Let U be again an open subset of δX. Define

$$\begin{aligned}C(U)=&\{z\in\bar{X}_{\mathbb{R}}:\exists\ \mathcal{O}\in\mathcal{N}(z)\text{ such that }\mathcal{O}\cap X\subseteq\mathcal{O}(U)\}\cup\\&\{z\in\delta X\backslash\bar{X}_{\mathbb{R}}:\exists\text{ open }U'\subseteq\delta X\text{ such that }z\in U'\text{ and }\mathcal{O}(U')\subseteq\mathcal{O}(U)\}.\end{aligned}$$

Now for $y\in\delta X\backslash\bar{X}_{\mathbb{R}}$ let

$$\mathcal{N}(y)=\{\mathcal{O}\subseteq\hat{X}:\exists\text{ open set }U\subseteq\delta X\text{ containing }y\text{ with }C(U)\subseteq\mathcal{O}\}.$$

This defines a system of neighborhoods $\mathcal{N}(y)$ for any $y\in\hat{X}$. For a subset $S\subseteq\hat{X}$ let $\mathcal{N}(S)=\{\mathcal{O}\subseteq\hat{X}:\mathcal{O}\in\mathcal{N}(y)\text{ for every }y\in S\}$ and call S *primary open* if $S\in\mathcal{N}(S)$. The following is again easy to check.

Proposition 3.9. *The primary open subsets of $\hat{X}$ form a topology.*

Definition 3.10. Let $\hat{X}_1$ be the set $\hat{X}$ with the *primary* topology.

Remark 3.11. The primary topology on $\hat{X}_1$ is not Hausdorff. This has to do with the rank of SL_3. Recall that each maximal 2-dimensional flat consists of six chambers and six walls. Pick two walls which are in opposition: they lie on a geodesic γ through the base point and determine two walls $W(P_1)$, $W(P_2)$ at infinity. If $z_1=q_{P_1}(\gamma)\in e(P_1)$ then let $z_1^u\in R_uP_1(\mathbb{R})$ be the first coordinate projection of $F^{-1}(z)$ (in the notation of Proposition 2.8). The point $z_2^u\in R_uP_2(\mathbb{R})$ is defined similarly. The two points are the limits of γ in $\bar{X}$. It turns out that the points of $\{z_1^u\}\times\hat{e}(P_1)$ and $\{z_2^u\}\times\hat{e}(P_2)$ match bijectively in this manner.

By Proposition 3.3 the principal $R_uP(\mathbb{R})$-fibration μ_P extends to $\bar{\mu}_{P,\mathbb{R}}$. Since each level is compactified as the hyperbolic disk in the beginning of the section, $\{z_i^u\}\times\overline{\hat{e}(P_i)}$, $i=1,2$, embed in the closures of the corresponding strata. It is now easy to see that the bijective correspondence described above extends to these enlargements and to find points $y_i \in \{z_i^u\} \times \big(\overline{\hat{e}(P_i)} - \hat{e}(P_i)\big)$ so that any two neighborhoods of y_1 and y_2 in the respective enlargements contain some points $x_i \in \{z_i^u\} \times \hat{e}(P_i)$ which are matched. Equivalently, y_1 and y_2 are inseparable in $\hat{X}_1$.

There is the obvious set projection $p\colon \hat{X} \longrightarrow X^S$ extending μ from Definition 2.10. This map should be identity on X and contract each conjugate of $\varepsilon(P_0)$. The complement consists of the conjugates of $\varepsilon(P_1)$ and $\varepsilon(P_2)$ which are projected onto the corresponding conjugates of the strata $\hat{e}(P_1)$ and $\hat{e}(P_2)$ fiberwise.

Definition 3.12. The *secondary* topology on $\hat{X}$ is the p-pull-back of the topology on X^S. Let $\hat{X}_2$ be the resulting topological space.

The secondary topology is again non-Hausdorff. By the product of two topologies on a set we mean the one generated by the union of bases for each topology.

Definition 3.13. Let $\hat{X}$ be the space topologized by the product of the primary and secondary topologies on the set $\hat{X}$.

4. Properties of the Compactification

Hausdorff property. For x_1, $x_2 \in \hat{X}$, if $p(x_1) = p(x_2) \in X^S$ then either x_1, $x_2 \in p^{-1}(y)$ for some $y \in X^S - X$ or $x_1 = x_2 \in X$. Now each $p^{-1}(y)$ is Hausdorff, so the points are separated in the primary topology. If $p(x_1) \neq p(x_2) \in X^S$ then the points are separated in the secondary topology.

Calculus of flats. In order to determine the geometry of open sets in $\hat{X}$, we need to study the geometric question: *describe the family of flats asymptotic to the given two chambers or walls at infinity of a symmetric space X*. The answer is quite natural in terms of horocycles.

Theorem 4.1 (Im Hof [13]). *If $y, z \in \partial X$ are contained in Weyl chambers $W(y)$, $W(z) \subseteq \partial X$, let N_y, N_z be the nilpotent components in the corresponding Iwasawa decompositions. For an arbitrary point $x \in X$ the intersection of the horocycles $N_y \cdot x \cap N_z \cdot x$ parametrizes the set of all flats asymptotic to both $W(y)$ and $W(z)$.*

Now the minimal strata $e(B)$ for $B \in \mathcal{B}_{\mathbb{R}}$ parametrize the flats which are asymptotic to $W(B)$.

Definition 4.2. Define the subsets $\mathcal{A}(B, B') \subseteq e(B)$ to be the geodesic projections $q_B(N_B \cdot x \cap N_{B'} \cdot x)$ in the sense that they consist of $a \in e(B)$ such that the flat $q_B^{-1}(a)$ is asymptotic to $W(B')$.

This parametrization is more convenient for us because each $\xi \in e(B)$ is precisely the point of intersection $e(B) \cap q_B^{-1}(\xi) = \{\xi\}$. Now given an open subset $U \subseteq e(B)$, the corresponding open set $C(U) \subseteq \hat{X}$ can be described as $\complement(\mathrm{cl}\, q_B^{-1}(\complement U))$, and $q_B^{-1}(\complement U)$ can be identified once the closure of each flat $q_B^{-1}(\xi)$, $\xi \notin U$, is known. This is easy to do for $X = X(SL_3)$.

Let $W(B)$, $W(B')$ be two adjacent Weyl chambers at infinity in $\partial X(SL_3)$, i.e., there exists $P \in \mathcal{P}_{\mathbb{R}} \backslash \mathcal{B}_{\mathbb{R}}$ with B, $B' \subseteq P$. Using the product structure on $\bar{X}_{\mathbb{R}}(P)$

and the fact that $\gamma_\xi = e(P) \cap \mathrm{cl}\big(q_B^{-1}(\xi)\big)$ are geodesics for each $\xi \in e(B)$, we see that the collection of flats which are asymptotic to both $W(B)$ and $W(B')$ is parametrized by $\mathcal{A}(B,B') = \bar{\mu}_{P,\mathbb{R}}^{-1}(y) \subseteq e(B)$, where $y \in e(B^P)$ is the endpoint of the well-defined hyperbolic geodesic $p(\gamma_\xi) \subseteq f(P)$ connecting $f(B)$ with $f(B')$. In other words, $\mathcal{A}(B,B')$ is precisely the set of such $\xi \in e(B)$.

Now let $W(B)$, $W(B')$ be two Weyl chambers at infinity which are neither adjacent nor in opposition (that is, do not contain ideal points represented by the opposite orientations of the same geodesic). This determines unique chamber $W(B')$ and walls $W(P)$, $W(P')$ with $B' = P \cap P'$, $B < P$, $B'' < P'$. So each flat asymptotic to $W(B)$ and $W(B'')$ will be also asymptotic to $W(B')$, and $\mathcal{A}(B,B'') \subseteq \mathcal{A}(B,B')$. Indeed, $\mathcal{A}(B,B')$ are points in the plane $\bar{\mu}_{P,\mathbb{R}}^{-1}(y)$. Let z be the other end of the geodesic $p(\gamma_\xi)$. There is a bijective correspondence

$$\mathcal{A}(B',B) = \bar{\mu}_{P,\mathbb{R}}^{-1}(z) \leftrightsquigarrow \mathcal{A}(B,B').$$

On the other hand, $\{z\} = f(P) \cap f(P')$, and

$$\mathcal{A}(B',B'') = \bar{\mu}_{P',\mathbb{R}}^{-1}(z) \leftrightsquigarrow \mathcal{A}(B'',B').$$

Now $\bar{\mu}_{P,\mathbb{R}}^{-1}(z)$ and $\bar{\mu}_{P',\mathbb{R}}^{-1}(z)$ are two transverse planes in $e(B')$ intersecting in a line L. It is clear that there are bijections

$$\mathcal{A}(B,B'') \leftrightsquigarrow L \leftrightsquigarrow \mathcal{A}(B'',B).$$

In terms of coordinates (x,y,z) in $e(B)$ induced from $e(P_0)$, $\mathcal{A}(B,B'')$ is a line $(x,y,*)$ with one of the coordinates x or y determined by the choice of $W(B')$ adjacent to $W(B)$, the other—by the choice of $W(B'')$ adjacent to $W(B')$.

Similarly, on one hand, each point in $\mathcal{A}(B,B'')$ uniquely determines a chamber $W(B^{(3)})$ adjacent to $W(B'')$, on the other—it determines a unique flat which is, therefore, the unique flat asymptotic to both $W(B)$ and $W(B^{(3)})$. This verifies

Proposition 4.3. *Given B, $B' \in \mathcal{B}_{\mathbb{R}}(SL_3)$, the flats which are asymptotic to both $W(B)$ and $W(B')$ are parametrized by*

$$\mathcal{A}(B,B') \overset{\sigma}{\leftrightsquigarrow} \mathcal{A}(B',B).$$

If $S \subseteq \mathcal{A}(B,B')$ then $\sigma(S) \subseteq \mathcal{A}(B',B)$ is contained in the closure $\mathrm{cl}\big(q_B^{-1}(S)\big)$.

In view of the discussion above, using the product structures in $\hat{X}(P)$, we can determine the geometry of $C(U)$ for an open subset $U \subseteq \varepsilon(B)$.

Proposition 4.4. *If B, $B' \in \mathcal{B}_{\mathbb{R}}(SL_3)$ and $U \subseteq \varepsilon(B)$ is an open subset then $y \in \varepsilon(B')$ is contained in $C(U)$ if and only if either*

(1) *$y \in e(B')$ and its orthogonal projection π_B onto $\mathcal{A}(B',B)$ is not contained in the subset $A_B(U)$ corresponding bijectively to $U \cap \mathcal{A}(B,B')$, or*

(2) *$y \in \partial e(B')$ and $y \notin \mathrm{cl}\big(\pi_B^{-1} A_B(U)\big)$.*

The intersections of $C(U)$ with $\varepsilon(P)$, $P \in \mathcal{P}_{\mathbb{R}} \backslash \mathcal{B}_{\mathbb{R}}$, are the obvious open product subsets.

From this description easily follows

Proposition 4.5 (Weak Summability). *Given arbitrary open subsets U_1 and $U_2 \subseteq \varepsilon(B)$, it may not be true that*

$$C(U_1 \cup U_2) = C(U_1) \cup C(U_2).$$

However, the open stars in any derived decomposition of $\varepsilon(B)$ *from Definition 3.2 do have this property.*

Corollary 4.6. *Given a finite collection of open subsets* $\Omega_1, \ldots, \Omega_n \subseteq \hat{X}$ *with* $\varepsilon(B) \subseteq \bigcup \Omega_i$ *there is another finite collection of open subsets* $U_1, \ldots, U_m \subseteq \varepsilon(B)$ *so that*

- $\varepsilon(B) \subseteq \bigcup U_j$,
- $\forall\ 1 \leq j \leq m\ \exists\ 1 \leq i \leq n$ *with* $C(U_j) \subseteq \Omega_i$,
- $C(\bigcup U_j) = \bigcup C(U_j)$.

Compactness. It can be shown that $\hat{X}_1$ is compact using [8] where summability was used implicitly as it holds obviously in the rank one case. However compactness of $\hat{X}_1$ and $\hat{X}_2$ alone does not imply compactness of $\hat{X}$. This will follow from

Lemma 4.7. *For each* $y \in X^S - X$ *and any open neighborhood* U *of* $\pi^{-1}(y)$ *in* $\hat{X}$ *there exists an open neighborhood* V *of* y *such that* $\pi^{-1}(y) \subseteq U$.

Proof. The topology in X^S can be described by making a sequence *convergent* if and only if it converges to a maximal flat and its projection onto the flat converges in Taylor's polyhedral compactification [12, 20].

Suppose $y = f(B)$ for some $B \in \mathcal{B}_{\mathbb{R}}$. Then U is a neighborhood of $\varepsilon(B) \subseteq \hat{X}$. We know from Corollary 4.6 that there is a neighborhood M of $\partial e(B) \subseteq \varepsilon(B)$ and a section $\sigma \colon e(B) \to X$ of q_B so that

$$N(B) := C(M) \cup W(e(B), \sigma, 0) \subseteq U,$$

and the closure of the complement of $N(B)$ does not intersect $\varepsilon(B)$. In particular, this means that for each flat F asymptotic to $W(B)$,

$$W(B) \notin \overline{F \cap N(B)}.$$

If λ_i is a sequence in $\pi(\overline{\complement N(B)}) \subseteq X^S$ converging to $f(B)$ then there is a sequence ϕ_i with the same limit contained in a flat asymptotic to $f(B)$. The preceding discussion shows that ϕ_i would lift to a sequence (ϕ_i itself!) converging to $W(B)$ which is impossible. So the closed set

$$\pi(\overline{\complement N(B)}) \cap f(B) = \emptyset.$$

Now any neighborhood of $f(B)$ in $\complement\pi(\overline{\complement N(B)})$ will do as V. A simpler argument works for $y \in \operatorname{int} f(P_i)$, $i = 1$ or 2. $\triangledown$

Let $\mathcal{U}$ be an arbitrary open covering of $\hat{X}$. Since $\pi^{-1}(y)$ is compact for each $y \in X^S$, let $U_{y,1}, \ldots, U_{y,n_y}$ be a finite collection of elements of $\mathcal{U}$ with

$$\pi^{-1}(y) \subseteq \bigcup_{i=1}^{n_y} U_{y,i}.$$

By Lemma 4.7 there is V_y such that

$$\pi^{-1}(V_y) \subseteq \bigcup_{i=1}^{n_y} U_{y,i}.$$

By compactness of X^S there is a finite collection of points $y_1, \dots, y_k$ with $X^S = V_{y_1} \cup \cdots \cup V_{y_k}$. Then

$$\hat{X} = \bigcup_{i=1}^{n_{y_1}} U_{y_1,i} \cup \cdots \cup \bigcup_{i=1}^{n_{y_k}} U_{y_k,i}.$$

Čech-acyclicity.

Definition 4.8. The modified *Čech homology* of a space Z with coefficients in S is the simplicial spectrum valued functor

$$\check{h}(Z;S) = \underset{\overleftarrow{CovZ}}{\operatorname{holim}} (N\_ \wedge S),$$

where $Cov Z$ is the category of finite rigid open coverings of Z defined in [3]. This is a generalized Steenrod homology theory.

Thus $\hat{X}$ being *Čech-acyclic* is equivalent to weak triviality of the homotopy inverse limit

$$\check{h}(\hat{X};KR) = \underset{\overleftarrow{\mathcal{U}\in Cov\hat{X}}}{\operatorname{holim}} (N\mathcal{U} \wedge KR).$$

Theorem 4.9. *If $f\colon X \to Y$ is a surjective continuous map, where Y and $f^{-1}(y)$ are Chogoshvili-acyclic for each $y \in Y$ and for any abelian coefficient group, then*

$$\check{f}\colon \check{h}(X;KR) \longrightarrow \check{h}(Y;KR)$$

is a weak homotopy equivalence. So both X and Y are Čech-acyclic.

Proof. Apply the weak Vietoris–Begle theorem for the modified Čech theory [8, Theorem 11.3.1]. The fibers need only be Chogoshvili-acyclic for the result of Inassaridze used in that proof. $\triangledown$

Now the fibers of $p\colon \hat{X} \to X^S$ are either points, disks, or closures of maximal Borel–Serre strata which are all Chogoshvili-acyclic by the theorem of Inassaridze and induction. Since X^S is Chogoshvili-acyclic by Theorem 2.11, Theorem 4.9 applies to p, and $\hat{X}$ is Čech-acyclic.

5. Proof of the Theorem

The general plan of the proof is common with [3, 4, 8, 9] which is to interpret the assembly map α as the Γ-fixed point map between two Γ-spectra

$$\begin{tikzcd} B\Gamma_+ \wedge K(R) \arrow[r, "\alpha(\Gamma)"] \arrow[d, "\simeq"'] & K(R\Gamma) \arrow[d, "\simeq"'] \\ \mathcal{R}^\Gamma \arrow[r, "\pi_*^\Gamma"] & \mathcal{T}^\Gamma \end{tikzcd}$$

Here $\mathcal{R}$ is the locally finite homology of $\bar{X}_{\mathbb{Q}}$ with coefficients in $K(R)$ and $\mathcal{T}$ is the Čech homology spectrum $\Sigma\check{h}(Y;K(R))$. We refer to [4] for the proof of the two equivalences in the diagram whenever Y is a boundary of the universal free Γ-space $E\Gamma$ satisfying a list of required properties. In our situation, $\bar{X}_{\mathbb{Q}}$ serves as a model for $E\Gamma$, Y is the boundary in the compactification from section 3, and all of the required properties were verified in section 4. The fixed point set map induces a map on homotopy fixed points and the following commutative square

$$\begin{tikzcd} \mathcal{R}^{\Gamma} \arrow[r, "\pi_*^{\Gamma}"] \arrow[d, "\simeq"'] & \mathcal{T}^{\Gamma} \arrow[d] \\ \mathcal{R}^{h\Gamma} \arrow[r, "\pi_*^{h\Gamma}"] & \mathcal{T}^{h\Gamma} \end{tikzcd}$$

It is known that $\pi_*^{h\Gamma}$ is a weak equivalence whenever $\pi_* \colon R \to \mathcal{T}$ is a weak equivalence. This makes the assembly map $\alpha(\Gamma)$ the first map in a composition which is a weak equivalence. It is then a split injection at the level of homotopy groups.

In fact, this choice of the target $\mathcal{T}$ was fine enough only in [3]. In other cited references the choice of the target had to be more refined. The idea is to replace

$$\check{h}(Y; K(R)) = \underset{\overleftarrow{Cov\,Y}}{\operatorname{holim}}\, N\_ \wedge K(R)$$

with a different homotopy limit over an equivariant category of covering sets in the boundary Y so that there is a weak equivalence

$$\theta \colon \underset{\overleftarrow{Cov\,Y}}{\operatorname{holim}}\, N\_ \wedge K(R) \longrightarrow \underset{\overleftarrow{A \in \mathcal{A}}}{\operatorname{holim}}\, NA \wedge K(R)$$

In order for the limit to fit into the commutative diagram and serve as a new target $\mathcal{T}$, the category $\mathcal{A}$ needs to satisfy a list of new conditions that we extract from [8, 9].

Definition 5.1. For any subset K of a metric space (X, d) let $K[D]$ denote the set $\{x \in X : d(x, K) \leq D\}$. If (X, d) is embedded in a topological space $\hat{X}$ as an open dense subset, a set $A \subseteq Y = \hat{X} - X$ is *boundedly saturated* if for every closed subset C of $\hat{X}$ with $C \cap Y \subseteq A$, the closure of each D-neighborhood of $C \setminus Y$ for $D \geq 0$ satisfies $\overline{(C \setminus Y)[D]} \cap Y \subseteq A$.

The required conditions on $\mathcal{A}$ are as follows.

(1) There is a subcategory $\mathcal{O}rd\,Y$ of $Cov\,Y$ such that the inclusion $\jmath \colon \mathcal{O}rd\,Y \hookrightarrow Cov\,Y$ induces a weak homotopy equivalence;
(2) For each set $U = \phi(y)$ for $\phi \in \mathcal{O}rd\,Y$ there is an open set $V(U) \subseteq \hat{X}$ with the following properties: (1) $V \cap Y = U$ and (2) $\{V(U) : U \in \operatorname{im}\phi\}_{\mathcal{O}rd\,Y}$ form a cofinal system of finite coverings of Y by open subsets of $\hat{X}$;
(3) Given a covering $\phi \in \mathcal{O}rd\,Y$, there is an assignment (which we call *saturation* and denote by **sat**) of a based boundedly saturated subset $A_y \subseteq Y$ to each set $\phi(y)$ so that **sat** induces a natural transformation

$$\mathbf{sat}_* \colon N\_ \wedge K(R) \longrightarrow N\mathbf{sat}(\_) \wedge K(R),$$

and the collection $\mathcal{A}$ above is precisely the result of applying saturation to $\mathcal{O}rd\,Y$. We require the resulting collection to be *excisive* in the sense defined in [4]. We require that each morphism $\mathbf{sat}_*$ is a weak equivalence of spectra by Quillen's Theorem A applied to $\mathbf{sat}_* \colon N\_ \to N\mathbf{sat}(\_)$.

The rest of the proof consists of the construction of coverings of Y satisfying the listed conditions.

Partial Cellular Decomposition. The cells in ∂N_0 described in Definition 3.2 make a symmetric picture in the standard corner where $N_0 \cong e(P_0)$ is the base of two geodesic bundles $e(P_1)$ and $e(P_2)$. There is a product structure in each $\varepsilon(P)$, $P \in \mathcal{P}_{\mathbb{Q}} \backslash \mathcal{B}_{\mathbb{Q}}$, so that the closure of each lift of $\hat{e}(P)$ in $e(P)\hat{}$ determines a point in the appropriate meridian $\sigma_{j,B}$ (depending on the type j of P) for each $B < P$. In $e(P_j)\hat{}$ the lift to $\xi_i \times \hat{e}(P_j)$ determines a vertex in σ_{j,P_0}. The complements in each of the two meridians are the 1-cells, and the complements of the circles in ∂N_0 are the 2-cells.

Similar decompositions are well-defined in other boundaries of Borel strata: for each $B \in \mathcal{B}_{\mathbb{R}}$ with $B < P'$, P'', $e(B) \cong R_u B(\mathbb{R})$ where $R_u P'(\mathbb{R})$ and $R_u P''(\mathbb{R})$ are subgroups isometrically embedded in the transported flat metric. Their intersection is a geodesic converging to the two 0-cells and $\partial R_u P'(\mathbb{R}) \cup \partial R_u P''(\mathbb{R})$ is the 1-skeleton in $\partial R_u B(\mathbb{R})$. The 2-cells are the four connected components of the complement. Now the derived cell structures in $\varepsilon(B)$ may be introduced so that the induced deriveds in $\partial R_u P(\mathbb{R})$ are compatible with the product structures in $\partial R_u P(\mathbb{R}) \times \varepsilon(\hat{e}(P))$.

Consider derived cubical decompositions of the unit square I^2 and their images under homeomorphisms $\pi_P \colon I^2 \to \varepsilon(\hat{e}(P))$. The nerves of the open star coverings of $\varepsilon(\hat{e}(P))$ or $\hat{e}(P)$ are clearly contractible. We consider finite open coverings $\mathcal{O}_{m,k,P}$ of $\varepsilon(P)$ by the products of open stars in (unrelated) cubical decompositions of $\partial R_u P(\mathbb{R})$ and $\hat{e}(P)$.

Let $\xi_{i,B}$ and $\vartheta^{(t)}_{j,B}$ for $i, j, t \in \{1, 2\}$ be the two 0-cells and four 1-cells in $\partial e(B)$ (here t is the type of the adjacent maximal parabolic subgroup). Define

$$\varsigma_{i,j,P} = \left(\xi_{i,B} \cup \vartheta^{(t)}_{i,B}\right) \times (\hat{e}(P))^{S}_{\mathbb{Q}},$$

where P is the maximal parabolic subgroup of type k containing B. It is clear that this definition is independent of the choice of $B < P$.

Bounded saturation. The space $\hat{X}$ contains $\bar{X}$ as an open dense Γ-subset; in particular, Γ acts continuously on $\bar{X}$ as before.

Definition 5.2. The metric that we use in $\bar{X}$ is a transported Γ-invariant metric. It is obtained by first introducing any bounded metric in the compact $\bar{X}/\Gamma$ and then taking the metric in $\bar{X}$ to be the induced path metric where the measured path-lengths are the lengths of the images in $\bar{X}/\Gamma$ under the covering projection.

Now we can identify a Boolean algebra of boundedly saturated sets fine enough for our purposes.

Proposition 5.3. *The following subsets of Y are boundedly saturated:*

- *each subset $\varsigma_{i,j,P}$ for $P \in \mathcal{P}_{\mathbb{Q}} \backslash \mathcal{B}_{\mathbb{Q}}$,*
- *each 2-cell in $\partial e(B)$ for $B \in \mathcal{B}_{\mathbb{Q}}$,*
- *each $\varepsilon(P)$ for $P \in \mathcal{P}_{\mathbb{R}} \backslash (\mathcal{P}_{\mathbb{Q}} \cup \mathcal{P}_{\mathbb{R}})$.*

Proof. For the purposes here, one can use commensurability inavariance of the saturation property [8] and substitute the given group Γ with $SL_3(\mathbb{Z})$ and use Grenier's fundamental domain. A union $\mathcal{T}$ of this domain and a finite number of its adjacent translates may be taken to intersect $e(P_0)$ in the domain $\mathcal{D}$ for the discrete Heisenberg group action. If ω is chosen to contain the domain $\mathcal{D}$ then the Siegel set Σ from Theorem 2.3 can be chosen (taking $t \geq 2/\sqrt{3}$) to contain $\mathcal{T}$. In fact, the corresponding domain and set for the action of the torsion-free Γ is a union of appropriate translates of $\mathcal{T}$ and Σ respectively. Now the domains are arranged so that one can isolate the candidates for boundedly saturated sets using "barriers" literally as in section 8.4 of [8]. $\triangledown$

Definition 5.4. The boundedly saturated sets identified in Proposition 5.3 generate a Boolean algebra of sets BA.

Orderly coverings. We will construct a cofinal family of finite open coverings of Y. Recall that a rigid covering $\beta \in CovY$ of Y consists of pairs $x \in U(x)$ where $x \in Y$ and the values $U(x)$ lie in a finite open covering of Y. Let $\mathcal{U}$ be the underlying finite open covering $\operatorname{im}\beta$.

Fix a Borel subgroup $B \in \mathcal{B}_{\mathbb{R}}(SL_3)$. There is a number $\ell_B \in \mathbb{N}$ with $\ell_B \geq n_B$ and an open neighborhood $U_B \ni f(B)$ in X^S with

$$\operatorname{PreInf}_{\ell_B,U_B}(v) \overset{\text{def}}{=} Y \cap C(\operatorname{Star}^{\text{o}}_{\ell_B}(v)) \cap p^{-1}U_B \subseteq \beta(x)$$

for each $v \in V_{(\ell_B)}$ and some $x \in Y$. Let $\mathfrak{I}$ be the set consisting of all $P \in \mathcal{P}_{\mathbb{R}} \setminus \mathcal{B}_{\mathbb{R}}$ such that $f(P) \cap \complement U_B \neq \emptyset$. Let $\mathfrak{F}$ consist of all $B' \in \mathcal{B}_{\mathbb{R}}$ such that

$$\mathcal{A}(B,P) \cap \operatorname{Star}^{\text{o}}_{\ell_B}(v) = \emptyset \quad \text{and} \quad \mathcal{A}(B,P) \cap \overline{\operatorname{Star}^{\text{o}}_{\ell_B}(v)} \neq \emptyset.$$

Now we can define $V_B(U) \subseteq U_B$ such that

$$U_B \backslash V_B = U_B \cap \bigcup_{B \not< P \in \mathfrak{I}} \overline{f(P)}$$

and

$$\operatorname{Inf}_{\ell_B,U_B}(v) \overset{\text{def}}{=} \operatorname{PreInf}_{\ell_B,U_B}(v) \cap p^{-1}(V_B) \backslash \bigcup_{B' \in \mathfrak{F}} \varepsilon(B').$$

The union of these sets over all $v \in V_{(\ell_B)}$ is an open neighborhood of $\varepsilon(B)$ in Y by the weak summability property.

Using compactness of $\hat{X}$, compactness of each $e(P)\hat{}$, $P \in \mathcal{P}_{\mathbb{R}}(SL_3)$, and relative compactness of $\varepsilon(P)$, one can choose finite subsets $\mathfrak{B} \subseteq \mathcal{B}_{\mathbb{R}}$ and $\mathfrak{P} \subseteq \mathcal{P}_{\mathbb{R}} \backslash \mathcal{B}_{\mathbb{R}}$ and numbers $0 < m_P$, $k_P \in \mathbb{N}$ for $P \in \mathfrak{P}$ satisfying

(1) $\forall B \in \mathfrak{B}\ \exists P \in \mathfrak{P}$ such that $B < P$,
(2) $Y = \bigcup_{B \in \mathfrak{B}} \operatorname{Inf}_{\ell_B,U_B}(v) \cup \bigcup_{P \in \mathfrak{P}} \varepsilon(P)$,

and the following properties: fix $P \in \mathfrak{P}$ and use the notation $\mathfrak{B}(P) := \{B \in \mathfrak{B} : B < P\}$, then

(3) for some $0 < k_P \in \mathbb{N}$ and $w(P) \in \partial\hat{e}(P) \cap V_{(k)}$

$$Y \cap \delta(\hat{e}(P)) = \bigcup_{B \in \mathfrak{B}} \operatorname{Inf}_{\ell_B,U_B}(v) \cap p^{-1}\operatorname{Star}^{\text{o}}_{k_B}(w) \cap \delta(\hat{e}(P)),$$

(4) $\mathcal{O}_{m,k,P}$ refines the restriction of $\mathcal{U}$ to $\varepsilon(P)$,
(5) $m_P \geq \max_{B \in \mathfrak{B}(P)}(\ell_B)$, $k_P \geq \max_{B \in \mathfrak{B}(P)}(k_B)$,
(6) each open star in the associated k_P-th cubical derived decomposition of $\varepsilon(\hat{e}(P))$ contains at most one point from $\{W(B) : B \in \mathfrak{B}(P)\}$,
(7) for each $w \in \partial\hat{e}(P) \cap V_{(k_P)}$ there exists $B \in \mathfrak{B}$ such that

$$\text{either } W(B) \in \operatorname{Star}^{\text{o}}_{k_P}(w) \text{ or } p^{-1}(\operatorname{Star}^{\text{o}}_{k_P}(w)) \subseteq \bigcup_{v \in V_{(\ell_B)}} \operatorname{Inf}_{\ell_B,U_B}(v).$$

For a Borel subgroup $B(w)$ define

$$\mathrm{Ord}_{\ell_B,U_B,k_P}(v;w) \stackrel{\text{def}}{=} \left(\mathrm{Inf}_{\ell_B,U_B}(v)\backslash e(P)^\wedge\right) \cup p^{-1}\mathrm{Star}^{\mathrm{o}}_{k_P}(w)$$

and

$$\mathrm{ExcOrd}_{\ell_B,U_B,k_P}(v;w) \stackrel{\text{def}}{=} \mathrm{Ord}_{\ell_B,U_B,k_P}(v;w)\backslash \bigcup_{B<P'} e(P')^\wedge.$$

Consider the category *ExcOrdY* of finite open coverings by the sets (Exc)$\mathrm{Ord}_{\ell_B,U_B,k_P}(v;w)$ and $\mathcal{O}_{m,k,P}$ for all choices of $\beta, \mathfrak{B}, \mathfrak{P}$, etc., and generate all finite rigid coverings $\omega \in \mathcal{C}ov\, Y$ which satisfy

- $\mathrm{im}\, \omega \in \mathit{ExcOrdY}$,
- $\omega(y) = \mathrm{Ord}_{\ell_P,k}(v;w)$ for some $P \in \mathfrak{P}$ if and only if $y \in \varepsilon(P)$,
- if $y \in \varepsilon(B)$ for some $B \in \mathfrak{P}$ then

  $$\omega(y) = \mathrm{ExcOrd}_{\ell_{P(w)},k}(v;w)$$

  for some v where $\chi\big(W(B)\big) = \mathrm{Star}^{\mathrm{o}}_k(w)$ for a fixed finite rigid covering χ of $\varepsilon X(M_1^0)$ by open stars $\mathrm{Star}^{\mathrm{o}}_k(z)$, $z \in V_{(k)}$,
- $\omega(y) \in \mathcal{O}_{m,k}$ if $y \in \varepsilon(P)$.

The resulting coverings form a full subcategory $\textsc{PreOrd}Y \subseteq \mathcal{C}ov\, Y$. This procedure may look asymmetric as to the roles of maximal strata played in corners

$$\overline{X(B)} = e(P')^\wedge \cup e(P'')^\wedge$$

when $P', P'' \in \mathfrak{P}$ and $y \in \varepsilon(B)$: there is a choice of w and, hence, of particular $P^{(j)}$ involved here. The asymmetry disappears after the next step when one generates the smallest full subcategory $\mathcal{O}rd\, Y$ of $\mathcal{C}ov\, Y$ containing $\textsc{PreOrd}Y$ and closed under intersections.

The category $\mathcal{O}rd\, Y$ is not cofinal; however the map

$$\jmath^*\colon \check{h}(Y;KR) \longrightarrow \underset{\overleftarrow{\mathcal{O}rd\, Y}}{\mathrm{holim}}\,(N_- \wedge KR)$$

induced by the inclusion $\jmath\colon \mathcal{O}rd\, Y \hookrightarrow \mathcal{C}ov\, Y$ is a weak homotopy equivalence by Quillen's Theorem A, cf. [8].

Definition of $\mathcal{A}$. The boundedly saturated coverings we produce are outcomes of actual saturation with respect to some Boolean algebra of boundedly saturated sets. Saturation enlarges the sets in $\mathcal{O}rd\, Y$ using the chosen coverings α_B, $B \in \mathfrak{B}$, and π_P, $P \in \mathfrak{P}$. It suffices to present the construction of boundedly saturated coverings $\alpha(\omega, \alpha_B, \pi_P)$ based on generators $\omega \in \textsc{PreOrd}Y$.

Definition 5.5. For $B \in \mathcal{B}_{\mathbb{R}}(SL_3)$ use the notation $\alpha_{1,B}$ or $\alpha_{2,B}$ for the finite rigid covering of $\sigma_{1,B}$ or $\sigma_{2,B}$ respectively given by $\alpha_{i,B}(y) = \alpha_B(y) \cap \sigma_{i,B}$ for each $y \in \sigma_{i,B}$. The same formula associates $\alpha_{i,B}(y) \subseteq \sigma_{i,B}$ to each $y \in \partial e(B)$. For $P > B$ of type i, define $\Pi_{B,P}\colon \delta e(B) \to \mathrm{im}\, \pi_P$ by

$$\Pi_{B,P}(y) = \begin{cases} \alpha_{i,B}(y) \times \big(\hat{e}(P)\big)^S_{\mathbb{Q}} & \text{if } B \in \mathfrak{B}, \\ \alpha_{i,B'}(v) \times \big(\hat{e}(P)\big)^S_{\mathbb{Q}} & \text{otherwise,} \end{cases}$$

where $B' \in \mathcal{B}_{\mathbb{R}}$ and the vertices v, w are from $\omega(y) = \mathrm{ExcOrd}_{\ell_{B'(w)}, U_{B'}, k_P}(v; w)$. Now define

$$\alpha^{\mathrm{int}}(y) = \begin{cases} \pi_P(y) & \text{if } y \in \varepsilon(P),\ P \in \mathfrak{P} \cap \mathcal{P}_{\mathbb{Q}} \\ \omega(y) \backslash \varepsilon(B) \cup \Pi_{B,P}(y) & \text{if } y \in \varepsilon(B),\ B \in \mathfrak{B}, \\ \omega(y) \cup \Pi_{B,P(j)}(y) & \text{if } y \in \sigma_{j,B},\ B \notin \mathfrak{B}, \\ \omega(y) & \text{otherwise.} \end{cases}$$

The *saturation* of a subset S with respect to a Boolean algebra of sets is the union of elements of BA which intersect S nontrivially. Define $\alpha(\beta)$ as the finite rigid covering of Y by the saturations of sets S in $\alpha^{\mathrm{int}}(\beta)$ with respect to the Boolean algebra BA from Definition 5.4. The equivariant category $\mathcal{A}$ is the collection of all such α.

Each of the two steps in this construction preserves the homotopy type of the nerve of ω and α^{int}. Now the natural transformation $N\_ \to N\alpha^{\mathrm{int}}(\_) \to N\alpha(\_)$ is composed of homotopy equivalences. So

$$\underset{\overleftarrow{\mathcal{O}rd\,Y}}{\mathrm{holim}}\,(N\_ \wedge KR) \xrightarrow{\simeq} \underset{\overleftarrow{\mathcal{O}rd\,Y}}{\mathrm{holim}}\,(N\alpha^{\mathrm{int}}(\_) \wedge KR) \xrightarrow{\simeq} \underset{\overleftarrow{\mathcal{O}rd\,Y}}{\mathrm{holim}}\,(N\alpha(\_) \wedge KR).$$

This procedure also defines a left cofinal saturation functor $\mathbf{sat}\colon \mathcal{O}rd\,Y \to \mathcal{A}$ so that the induced map

$$\mathbf{sat}_*\colon \underset{\overleftarrow{\mathcal{O}rd\,Y}}{\mathrm{holim}}\,(N\alpha(\_) \wedge KR) \xrightarrow{\simeq} \underset{\overleftarrow{\mathcal{A}}}{\mathrm{holim}}\,(N\_ \wedge KR).$$

The composition of all weak equivalences above is the required weak equivalence

$$\theta\colon \check{h}(Y; KR) \simeq \underset{\overleftarrow{\mathcal{A}}}{\mathrm{holim}}\,(N\_ \wedge KR).$$

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