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Novikov Conjectures for Arithmetic Groups with Large Actions at Infinity

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Abstract. We construct a new compactification of a noncompact rank one globally symmetric space. The result is a nonmetrizable space which also compactifies the Borel–Serre enlargement $\bar{X}$ of X, contractible only in the appropriate Čech sense, and with the action of any arithmetic subgroup of the isometry group of X on $\bar{X}$ not being small at infinity. Nevertheless, we show that such a compactification can be used in the approach to Novikov conjectures developed recently by G. Carlsson and E. K. Pedersen. In particular, we study the nontrivial instance of the phenomenon of bounded saturation in the boundary of X and deduce that integral assembly maps split in the case of a torsion-free arithmetic subgroup of a semi-simple algebraic $\mathbb{Q}$-group of real rank one or, in fact, the fundamental group of any pinched hyperbolic manifold. Using a similar construction we also split assembly maps for neat subgroups of Hilbert modular groups.

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Introduction

Let Γ be a discrete group. Consider the assembly map $\alpha\colon B\Gamma_+ \wedge S(R) \to S(R\Gamma)$, where $S(R)$ is the K-theory spectrum for a ring R (see Section 1.3 below or Loday [41]). There are also L-, A-theoretic and C^*-algebraic versions of this map. It is known that for $S = L$ and $R = \mathbb{Z}$, the splitting of α implies the classical form of the Novikov conjecture on the homotopy invariance of higher signatures for manifolds with the fundamental group Γ. By analogy, each of the other versions is called the (*integral*) *Novikov conjecture in S-theory*, and there are separate reasons for proving each of them (see Section 1.3). In the presence of torsion, assembly maps do not always split, so attention is naturally restricted to torsion-free groups.

Carlsson, Pedersen, and Vogell verified the conjecture in K-, L- and A-theories for groups satisfying certain conditions ([16, 19]). For the sake of simplicity we state only the K-theoretic version:

THEOREM 1 (Carlsson–Pedersen). *Suppose there exists $E\Gamma$ such that the Γ-action is cocompact and extends to a contractible, metrizable compactification $\widehat{X}$ of $E\Gamma$ so that the action of Γ on $\widehat{X}$ is small at infinity, then α is a split injection.*

If Γ acts on a space X with an equivariant compactification $\widehat{X} = X \cup Y$, the action is called *small at infinity* if for every $y \in Y$, compact subset $Z \subset X$, and neighborhood $U \subset \widehat{X}$ of y, there exists a neighborhood $V \subset \widehat{X}$ of y such that if $gZ \cap V \neq \emptyset$ for some $g \in \Gamma$ then $gZ \subset U$. Examples of such situations are the ideal compactification of a complete nonpositively curved manifold (with a cocompact action) or the analogous constuction for a contractible Rips complex associated to a Gromov hyperbolic group. Notice that these are essentially geometric compactifications performed with one's eye toward extending (quasi-)isometries to the boundaries so that quasi-identities extend to trivial maps of the boundary. In particular, every subset of Y in these examples is boundedly saturated in the sense of the following definition. A set $A \subset Y$ is *boundedly saturated* if for every closed set C in $\widehat{X}$ with $C \cap Y \subset A$ the closure of any of the d-neighborhoods of $C \backslash Y$ satisfies $\overline{(C \backslash Y)[d]} \cap Y \subset A$.

After a considerable refinement of the methods in [17], this result has been improved to

THEOREM 2 (Carlsson–Pedersen). *Suppose there exists $E\Gamma$ with the one-point compactification $E\Gamma^+$ such that the Γ-action on $E\Gamma$ is cocompact and extends to a Čech-acyclic compactification $\widehat{X} = E\Gamma \sqcup Y$ so that there is a Γ-invariant system $\{\alpha\}$ of coverings of Y by boundedly saturated open sets and a weak homotopy equivalence*

$$\underset{\mathcal{U} \in \mathit{Cov}\, E\Gamma^+}{\underleftarrow{\operatorname{holim}}} (N\mathcal{U} \wedge KR) \simeq \Sigma \underset{\alpha \in \{\alpha\}}{\underleftarrow{\operatorname{holim}}} (N\alpha \wedge KR),$$

then α is a split injection.

This theorem is part of a very general approach initiated in [13, 15]. The statement of Theorem 2 and its modification that we actually use will be explained in more detail in Section 3. The purpose of this article is to provide examples where these new phenomena appear and get used. A general torsion-free arithmetic group seems to admit similar constructions, but then their analysis becomes more involved. From such a perspective, this paper completes the first two steps in a general inductive argument. Here we prove

THEOREM 3. *Let G be a semi-simple linear algebraic group defined over $\mathbb{Q}$ of real rank one. If Γ is a torsion-free arithmetic subgroup of G then α is a split injection.*

The simplest example of the situation in Theorem 3 is when $G = \mathrm{SL}_2$. Let X be the hyperbolic disk which is the symmetric space associated to G. Borel and Serre construct $E\Gamma$ for any torsion-free arithmetic subgroup Γ of G by blowing up each rational point on the boundary circle of X to a line. To compactify this space, we blow up all of the remaining points on the circle and provide each of the lines with two limit points. The resulting boundary of X is set-theoretically the cylinder $S^1 \times I$. The compact Hausdorff topology we introduce in our space restricts to the circular lexicographic order topology on $S^1 \times I$. In particular, for any arc $C \subset S^1$, $C \times I$ is homeomorphic to the classical unit square with the lexicographic order topology. Recall that this topology is compact but not separable. This makes $C \times I$ and $S^1 \times I$ and our space nonmetrizable.

It will be observed that there is an analogue of the construction above where only the rational points are resolved to lines. For some but not all groups in Theorem 3 the action at infinity in the altered compactification is small. For groups where a cusp stabilizer is non-Abelian (e.g., for symplectic groups) the action will not be small in either compactification; hence the title of this paper.

Many of our constructions and results can be done and hold in greater generality than needed for the proof of Theorem 3. For example, Section 4 compactifies $N = E\Gamma$ for a torsion-free finitely generated nilpotent group Γ. We could follow it by a proof of the Novikov conjecture for such groups which is not a new result by itself (cf. [15, 48]). The importance of Section 4 is the role as the base case it plays in the construction of $\widehat{X}$ for an arbitrary arithmetic group. Here the action of Γ on $\widehat{N}$ is already not small at infinity. This property is preserved in the ambient construction for G from Theorem 3 where copies of $\widehat{N}$ for certain one or two step nilpotent groups embed. Section A.3 contains a discussion of this situation and its relation to other approaches to Novikov conjectures.

The arithmeticity hypothesis in Theorem 3 can be dropped. When the construction of Borel–Serre in our argument is replaced by a 'neutered' pinched Hadamard manifold (as in [25]), the stabilizers of boundary components are nilpotent, and the proof of Theorem 3 in conjunction with Section 4 works verbatim to show

THEOREM 4. *If Γ is a torsion-free fundamental group of a complete noncompact finite-volume Riemannian manifold with pinched negative sectional curvatures $-a^2 \leqslant K \leqslant -b^2 < 0$ then α is a split injection.*

It is known from [35] that there are pinched hyperbolic manifolds in each dimension $n \geqslant 4$ which are not locally symmetric. All of these groups may be classified as hyperbolic relative to a finite family of nilpotent subgroups in the sense of [25]. It seems very plausible that using a cross of the constructions of Rips and Borel–Serre, our argument also applies in this combinatorial situation.

In Section A.2 the argument is adjusted slightly to apply to lattices in semi-simple Lie groups of higher $\mathbb{R}$-rank:

THEOREM 5. *If Γ is a neat arithmetic subgroup of a Hilbert modular group then α is a split injection.*

The use of a 'topological' approach as in Theorem 2 seems to be essential in both of our applications. Recall that no $\mathrm{SL}_2(\mathcal{O}_d)$ is bicombable ([31, Proposition 6.14]) and neither of the groups Γ in $G \neq \mathrm{SO}(n, 1)$ from Theorem 3 is combable ([24, Theorem 1.2]). These results make it doubtful that our groups have reasonable geometric compactifications with small actions at infinity for it is precisely the combings that are used to produce examples after Theorem 1.

The main body of this paper deals with the K-theoretic assembly map. In Section A.1 we discuss the extension of our results to other versions of the map.

The material in this paper is a part of my Cornell Ph.D. thesis [32] which also contains a proof of Novikov conjectures for torsion-free lattices in the semi-simple group SL_3 of split rank two.

1. Preliminary Material

1.1. HOMOTOPY LIMITS

We will use the language of simplicial homotopy theory ([43], [10, Part II]). A functor from a small category $F: \mathcal{C} \to \mathcal{D}$ is also called a $\mathcal{C}$-*diagram in* $\mathcal{D}$. Recall that the limit and the colimit of F are objects of $\mathcal{D}$ characterized by certain universal properties. They may not exist for an arbitrary diagram in S$-$SETS. The homotopy limit and colimit are simplicial sets which exist for any diagram F and satisfy universal properties with homotopy theoretic flavor. Homotopy limits are natural in both variables. Here is a list basic properties of homotopy limits and colimits which will be referred to later.

THEOREM 1.1.1 ([10, XI, Section 3]). *There are natural maps*

$$\underset{\overleftarrow{\mathcal{C}}}{\lim}\, F \longrightarrow \underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}\, F \quad \text{or} \quad \underset{\overrightarrow{\mathcal{C}}}{\mathrm{hocolim}}\, F \longrightarrow \underset{\overrightarrow{\mathcal{C}}}{\mathrm{colim}}\, F$$

whenever the appropriate limit or colimit exists.

THEOREM 1.1.2 (Homotopy Invariance, [10, XI, Section 5]). *Let $\phi: F \to G$ be a natural transformation of functors such that each $\phi(C): F(C) \to G(C)$, $C \in \mathcal{C}$, is a weak equivalence. Then* hocolimϕ *is a weak equivalence. If $F(C)$ and $G(C)$ are Kan for all $C \in \mathcal{C}$ then* holimϕ *is also a weak equivalence.*

THEOREM 1.1.3 (Cofinality Lemma, [10, XI, Section 9]). *Let $\Phi: \mathcal{C} \to \mathcal{C}'$ and $F: \mathcal{C}' \to$* S$-$SETS *be functors from small categories. If Φ is right cofinal (that is, $C' \downarrow \Phi$ is nonempty and contractible for every $C' \in \mathcal{C}'$) then* hocolimΦ *is a weak equivalence. If Φ is left cofinal (that is, $\Phi \downarrow C'$ is nonempty and contractible for every $C' \in \mathcal{C}'$) and each $F(C')$, $C' \in \mathcal{C}'$, is Kan, then* holimΦ *is a weak equivalence.*

THEOREM 1.1.4 ([21, Section 9]). *Let $\mathcal{C}$ be a contractible small category and $F: \mathcal{C} \to$* S$-$SETS *be a functor such that, for each morphism $c \in \mathcal{C}$, $F(c)$ is a weak equivalence. Then, for every object $C \in \mathcal{C}$, the obvious map*

$$\tau_F(C): FC \longrightarrow \underset{\overrightarrow{\mathcal{C}}}{\mathrm{hocolim}}\, F$$

is a weak equivalence. If each $F(C)$, $C \in \mathcal{C}$, is Kan then

$$\tau^F(C)\colon \underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}} F \longrightarrow FC$$

is a weak equivalence.

We assume familiarity with the language of spectra ([1]). The results above generalize to simplicial spectra ([15]), the notion of homotopy (co)limit being extended via a level-wise construction in the obvious way. The foundational material on simplicial spectra can be found in [53, Section 5].

THEOREM 1.1.5 (Bousfield–Kan Spectral Sequence [10, 15]). *Given a functor $F\colon \mathcal{C} \to$ SPECTRA, let $\pi_i \circ F\colon \mathcal{C} \to$ ABGROUPS be the composition with the stable π_i. Then there is a spectral sequence converging to*

$$\pi_*(\underset{\overleftarrow{\mathcal{C}}}{\operatorname{holim}} F) \quad \textit{with} \quad E_2^{p,q} = \underset{\overleftarrow{\mathcal{C}}}{\lim}^{\mathrm{p}} (\pi_q \circ F).$$

The following strengthening of the general Cofinality Lemma is very useful.

THEOREM 1.1.6 (Modified Cofinality Lemma [17, Lemma 2.8]). *Let $\mathcal{P}$ be a left filtering partially ordered set viewed as a category, and let $\imath\colon \mathcal{P}^0 \hookrightarrow \mathcal{P}$ be the inclusion of a partially ordered subset, also left filtering. Let $F\colon \mathcal{P} \longrightarrow$ SPECTRA be a functor and assume that for every $x \in \mathcal{P}$ there exist $x' \in \mathcal{P}$ and $y \in \mathcal{P}^0$ so that $x' > x$, $x' > y$, and so that $F(x' > y)$ is a weak equivalence. Then the restriction map*

$$\imath^*\colon \underset{\overleftarrow{\mathcal{P}}}{\operatorname{holim}} F \longrightarrow \underset{\overleftarrow{\mathcal{P}^0}}{\operatorname{holim}} F$$

is a weak equivalence.

1.2. ALGEBRAIC *K*-THEORY

This describes what we mean by K-theory here. In [47] Quillen constructed K-groups of a ring R, $K_n(R)$, $n \geqslant 0$. Before that the lower K-groups $K_n(R)$, $-\infty < n \leqslant 2$, were studied by Bass, Milnor, and others. The groups of Quillen can be obtained as stable homotopy groups of connective spectra. The most suitable delooping machine to use in this situation is Thomason's ([54]) functor Spt. Pedersen and Weibel ([44, 45]) used this functor and controlled algebra to produce a nonconnective spectrum $K(R)$ whose homotopy groups are all $K_n(R)$, $n \in \mathbb{Z}$. They also show that this agrees with the nonconnective spectrum of Gersten and Wagoner.

The functor of Thomason constructs a connective spectrum for every small symmetric monoidal category. If $\mathcal{F}$ is the symmetric monoidal category of isomorphisms of free finitely generated R-modules then $K^{\text{conn}}(R) = \text{Spt}(\mathcal{F})$ is the connective K-theory spectrum of R. More generally, if $\mathcal{A}$ is a small additive category then the category of isomorphisms $i\mathcal{A}$ of $\mathcal{A}$ is a symmetric monoidal category. Let $\mathcal{C}_k(\mathcal{A})$ denote the category of $\mathcal{A}$-objects parametrized over the metric space $\mathbb{Z}^k$ and bounded morphisms (the prototype of the categories defined in Section 7) then [45] constructs functorial maps

$$\text{Spt}(i\mathcal{C}_k(\mathcal{A})) \longrightarrow \Omega\text{Spt}(i\mathcal{C}_{k+1}(\mathcal{A})).$$

Taking

$$K(\mathcal{A}) = \underset{\overrightarrow{n>0}}{\text{hocolim}}\, \Omega^n \text{Spt}(i\mathcal{C}_n(\mathcal{A}))$$

one gets a nonconnective spectrum. Again, if $\mathcal{A}$ is the category of free finitely generated R-modules then $K(R) = K(\mathcal{A})$ is the Gersten–Wagoner spectrum.

1.3. ASSEMBLY IN ALGEBRAIC K-THEORY

Let Γ be a discrete group and R a ring. The assembly map in algebraic K-theory

$$\alpha_n \colon h_n(B\Gamma; KR) \to K_n(R\Gamma)$$

was first constructed by J.-L. Loday. Let $\imath\colon \Gamma \to \text{GL}_n(R\Gamma)$ be the inclusion of Γ in $(R\Gamma)^\times = \text{GL}_1(R\Gamma)$. Then there is a map

$$\Gamma \times \text{GL}_n(R) \xrightarrow{\imath\times\text{id}} \text{GL}_1(R\Gamma) \times \text{GL}_n(R) \xrightarrow{\otimes} \text{GL}_n(R\Gamma)$$

defined by

$$g, (a_{ij}) \longmapsto (g \cdot a_{ij}).$$

One can apply the classifying space functor B, pass to the limit as $n \to \infty$, and apply Quillen's plus construction to induce the map

$$B\Gamma_+ \wedge \text{BGL}(R)^+ \xrightarrow{B\imath^+ \wedge \text{id}} \text{BGL}(R\Gamma)^+ \wedge \text{BGL}(R)^+ \xrightarrow{\gamma} \text{BGL}(R\Gamma)^+.$$

This product is compatible with the infinite loop space structure of $\text{BGL}(\_)^+$ ([41, Section 11.2.16]). Delooping of this map results in the assembly map of spectra

$$\alpha \colon B\Gamma_+ \wedge K(R) \to K(R\Gamma),$$

where $B\Gamma_+$ is the classifying space together with a disjoint base point, and $K(R)$ is the Gersten–Wagoner nonconnective K-theory spectrum as in Section 1.2. This is the *assembly map in algebraic K-theory*. Loday's assembly map is induced by taking the homotopy groups:

$$h_n(B\Gamma; KR) = \pi_n(B\Gamma_+ \wedge KR) \xrightarrow{\alpha_n} \pi_n(K(R\Gamma)) = K_n(R\Gamma).$$

There is at least a couple of reasons why the study of this map is of importance in geometric topology. One is the involvement of $K(\mathbb{Z}\Gamma)$ in the description of the space of automorphisms of a manifold M with $\pi_1 M = \Gamma$. The other is the connection with Novikov and Borel conjectures.

It is known that the homotopy invariance of higher signatures follows from the splitting of the rational assembly map α in L-theory. The assembly naturally maps the rational group homology containing the signature to the surgery L-group where the image is a priori homotopy invariant. If the assembly is actually an injection then the signature is homotopy invariant. This is the modern approach to proving the Novikov conjecture. In fact, stronger integral conjectures can be stated when integral group homology is used, and there are K-, A-theoretic, and C^*-algebraic analogues of these integral maps. For example, the statement parallel to the above about classes in $KO[\frac{1}{2}]$ is equivalent to integral injectivity of α (see [57]). It makes sense, therefore, to call the conjecture that the assembly map in K-theory is injective for torsion-free Γ *the integral Novikov conjecture in K-theory*. A stronger and geometrically important conjecture that α is an isomorphism is then the K-theoretic part of the Borel conjecture. For example, the vanishing of $\mathrm{Wh}(\Gamma)$ would follow as a corollary to this.

There is another very interesting geometric application. The splitting of the C^*-algebraic version of the assembly map which can be obtained by applying the same approach as taken here (Carlsson–Pedersen–Roe) gives what J. Rosenberg calls the *strong Novikov conjecture*. That is known to imply rigidity and vanishing results for higher elliptic genera ([40]).

1.3. ARITHMETIC GROUPS

Let G be a linear algebraic group defined over $\mathbb{Q}$ and write $G(\mathbb{Z}) = G(\mathbb{Q}) \cap \mathrm{GL}_n(\mathbb{Z})$.

DEFINITION 1.3.1. A subgroup Γ of $G(\mathbb{Q})$ is *arithmetic* if Γ and $G(\mathbb{Z})$ are commensurable, that is, if the subgroup $\Gamma \cap G(\mathbb{Z})$ has finite index in both Γ and $G(\mathbb{Z})$. A discrete group Γ is *arithmetic* if it is isomorphic to an arithmetic subgroup of some group G as above.

Consider the real points $G(\mathbb{R})$ of G. It is a real Lie group, and $\Gamma \subset G(\mathbb{R})$ is a discrete subgroup. The $\mathbb{R}$-rank of G coincides with the rank of the symmetric space X associated with $G(\mathbb{R})$. When G is semi-simple, Γ acts freely and properly discontinuously on X. The quotient manifold $M = X/\Gamma$ is not compact unless $\mathrm{rank} G = 0$ but has finite invariant volume, i.e., Γ is a nonuniform lattice in $G(\mathbb{R})$.

The most famous class of arithmetic groups are *congruence subgroups* defined as the kernels of surjective maps $G(\mathbb{Z}) \to G(\mathbb{Z}_\ell)$ induced by reduction mod ℓ for various *levels* ℓ. Every arithmetic group contains a torsion-free subgroup of finite index, but, according to Minkowski, the congruence subgroups of SL_n of all levels $\ell \neq 2$ are themselves torsion-free (see [12, p. 40]).

If G is a connected linear simple Lie group with $\mathbb{R}$-rank one, there is a complete classification available ([55]). The four possibilities are the Lorentz groups $\mathrm{SO}_0(n,1)$, $\mathrm{SU}(n,1)$, $\mathrm{Sp}(n,1)$, and $\mathbb{F}_4$, the automorphism group of the exceptional simple Jordan algebra or, equivalently, the group of isometries of the Cayley projective plane with the appropriate Riemannian metric (see [2]). The class of rank one arithmetic groups contains representatives of various interesting group-theoretic phenomena: discrete subgroups of $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$ are K-amenable while lattices in $\mathrm{Sp}(n,1)$ have Kazhdan property T.

Examples of torsion-free arithmetic subgroups here can be congruence subgroups of level $\ell \geqslant 3$ of $\mathrm{SL}(n+1,\mathbb{Z}) \cap \mathrm{SO}_0(n,1)$. This identifies a particular system of torsion-free arithmetic groups to which our Theorem 3 applies.

2. Modified Čech Homology

This section explains the setup for the recent work of Gunnar Carlsson and Erik Pedersen referred to in the Introduction.

2.1. CLASSICAL ČECH HOMOLOGY

Let $\mathcal{U}$ be an open covering of a topological space X. The *nerve* $N\mathcal{U}$ of $\mathcal{U}$ is the simplicial complex with members of $\mathcal{U}$ as vertices and a simplex $\{U_1, \dots, U_s\}$ for each subset with $U_1 \cap \cdots \cap U_s \neq \emptyset$. We may think of $N\mathcal{U}$ as a simplicial set $N_\bullet \mathcal{U}$. If $\mathcal{V}$ is another open covering of X, and for each $U \in \mathcal{U}$ there is $V(U) \in \mathcal{V}$ so that $U \subset V(U)$ then one says that $\mathcal{U}$ *refines* $\mathcal{V}$ and writes $\mathcal{U} > \mathcal{V}$. If $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in \mathcal{B}}$ then the *map* of coverings $\mathcal{U} \to \mathcal{V}$ is a set map $f\colon \mathcal{A} \to \mathcal{B}$ such that $U_\alpha \subset V_{f(\alpha)}$ for all $\alpha \in \mathcal{A}$. Clearly, this map of vertices extends to a map of nerves $Nf\colon N\mathcal{U} \to N\mathcal{V}$. All such maps for one particular refinement $\mathcal{U} > \mathcal{V}$ are contiguous (see [23]) and, therefore, induce the same map on homology or homotopy groups of the nerves. Consider the partially ordered system $\mathrm{Cov}\, X$ of all finite open coverings of X. The resulting inverse system of Abelian groups $\{H_n(N\_; S)\}_{\mathrm{Cov}\, X}$ always produces the inverse limit $\check{H}_n(X; S)$ called the nth *Čech homology group*. The contiguity property of the refinement maps implies that the same inverse system is obtained as $\{H_n(N\_; S)\}_{\mathrm{Cov}^{\mathrm{s}} X}$, where $\mathrm{Cov}^{\mathrm{s}}\, X$ is the category of coverings and maps even though the functor itself is no longer a pro-Abelian group.

Given a map of spaces $f\colon X \to Y$, any $\mathcal{V} \in \mathrm{Cov}\, Y$ pulls back to a covering $f^*\mathcal{V} \in \mathrm{Cov}\, X$ in the obvious way. The injections $Nf^*\mathcal{V} \to N\mathcal{V}$ induce the universal map

$$\varprojlim_{f^* \mathrm{Cov}\, Y} H_n(N\_; S) \to \check{H}_n(Y; S).$$

But the inclusion of inverse systems $f^*\mathrm{Cov}\,Y \subset \mathrm{Cov}\,X$ induces another universal map

$$\check{H}_n(X;S) \to \lim_{\substack{\longleftarrow \\ f^*\mathrm{Cov}\,Y}} H_n(N\_;S)$$

and the composition of the two is denoted by f_*. This makes $\check{H}_*$ functors. Actually, $\check{H}_*$ are almost a homology theory: they do not satisfy the exactness axiom ([23]). This is the classical Čech homology theory.

2.2. MODIFIED ČECH HOMOLOGY

Another way to construct a functor similar in spirit is to take the inverse limit of the diagram of nerves $\{N\_\}_{\mathrm{Cov}^{\mathrm{s}}X}$, or spectra $\{N\_ \wedge S\}_{\mathrm{Cov}^{\mathrm{s}}X}$, or simplicial spectra $\{N_{\bullet}\_ \wedge S\}_{\mathrm{Cov}^{\mathrm{s}}X}$, and then take homology groups, or stable homotopy groups, of the result. However, for the limit above to always exist, it must be a homotopy inverse limit. Notice also that the functor $N\_$ can only be defined on $\mathrm{Cov}^{\mathrm{s}}\,X$ and not on $\mathrm{Cov}\,X$. The maps are induced just as above.

Notation. Whenever we write holim $(N\_ \wedge KR)$ we understand a simplicial spectrum, where $N\_$ stands for the simplicial set generated by the classical nerve complex via the total singular complex functor. The maps are usually induced from PL maps.

Remark 2.2.1. The values of the functor $N\colon \mathrm{Cov}^{\mathrm{s}}\,\widehat{X} \to$ S−SETS mapping a covering $\mathcal{U}$ to the simplicial nerve $N\mathcal{U}$ are not necessarily fibrant. To improve homotopy invariance properties of homotopy limits we adopt a convention which is used in [15]. Recall that there is a functorial replacement $K^\infty Q\colon \mathcal{S} \to \omega\mathcal{S}$ of a spectrum by a weakly equivalent Kan Ω-spectrum. The convention is that if $F\colon \mathcal{C} \to \mathcal{S}$ is a diagram whose values are not Kan Ω-spectra then the notation holim(F) will mean holim$(K^\infty QF)$. This convention simplifies hypotheses in standard results about homotopy limits.

Recall that $\mathcal{C}$ is a *left filtered* category if for any two objects $C_1, C_2 \in \mathcal{C}$ there exists $C_3 \in \mathcal{C}$ with $\mathrm{Mor}(C_3, C_1) \neq \emptyset \neq \mathrm{Mor}(C_3, C_2)$. If, in addition, for any two morphisms $m_1, m_2 \in \mathrm{Mor}(C, C')$ in $\mathcal{C}$ there exists $C' \in \mathcal{C}$ and $m \in \mathrm{Mor}(C', C)$ with $m_1 \circ m = m_2 \circ m$, then $\mathcal{C}$ is called *left filtering*. According to Quillen ([47]), every left filtering category is contractible.

Note that the homotopy limit above is taken over the category $\mathrm{Cov}^{\mathrm{s}}\,X$ with morphism sets $\mathrm{Mor}(\mathcal{U}, \mathcal{V})$ consisting of contiguous maps $\mathcal{U} \to \mathcal{V}$. It is easy to see that although $\mathrm{Cov}^{\mathrm{s}}\,X$ is left filtered, it is not left filtering. Instead of this category Carlsson and Pedersen use, following Friedlander, the category of *rigid coverings*. This category is, in fact, a partially ordered set: morphism sets are either empty or singletons. One advantage of this choice is the ease with which the Cofinality

Lemma 1.1.3 and the Modified Cofinality Lemma 1.1.6 can be applied. The more important consequence is the exactness property for the resulting Čech homology (Definition 2.2.3).

DEFINITION 2.2.2. A *finite rigid covering* of X is a set function β from X to open subsets of X which takes only finitely many values and satisfies (1) $x \in \beta x$ for all $x \in X$ and (2) $\overline{\beta^{-1} U} \subset U$ for all $U \in \operatorname{im}(\beta)$. Each finite rigid covering can be thought of as a covering in the usual sense. Set $N(\beta) = N(\{\beta(x) : x \in X\})$. This time the nerve is an infinite simplicial complex unless X is finite.

We will denote the category of finite rigid coverings by $Cov\, X$. There is a unique map $\beta_1 \to \beta_2$ if $\beta_1(x) \subset \beta_2(x)$ for all $x \in X$. Now $Cov\, X$ is left filtering, so the maps can be indicated simply: $\beta_1 > \beta_2$.

Define $F\colon Cov\ X \to Cov^{\mathrm{s}} X$ to be the forgetful functor $\beta \mapsto \{U_x = \beta(x)\}_{x \in X}$, where $Cov^{\mathrm{s}} X$ is the category of open coverings of X which may be infinite as sets but employ only finitely many open subsets of X. In particular, $F\beta$, $\beta \in Cov\ X$, are always infinite if X is infinite, but the covering sets come from the finite $\operatorname{im} \beta$. Now $N(\beta) = N(F(\beta))$ is clearly a functorial construction.

Let us emphasize that the assignment $\beta \mapsto \operatorname{im} \beta \in \mathrm{Cov}^{\mathrm{s}}\, X$ is *not functorial*. However, the obvious projection $F(\beta) \to \operatorname{im} \beta$ induces a homotopy equivalence on nerves according to Quillen's Theorem A.

DEFINITION 2.2.3. The *Čech homology* of X with coefficients in S is the simplicial spectrum valued functor

$$\check{h}(X; S) = \underset{\overleftarrow{Cov\ X}}{\operatorname{holim}} (N\_ \wedge S).$$

THEOREM 2.2.4 ([17]). *$\check{h}(\_\,; S)$ is a Steenrod homology theory.*

OTHER MODIFICATIONS

OthMods The construction of the modified Čech homology is almost what Edwards and Hastings did in [22, Section 8.2] to construct their Steenrod extension ${}^s h(X; S)$. They used the functor $V\colon \mathrm{TOP} \to \mathrm{PRO{-}S{-}SETS}, X \mapsto \{VN(\mathcal{U}) : \mathcal{U}$ an open cover of $X\}$, where VN denotes the Vietoris nerve. The rigidity of the Vietoris construction makes V land in a pro-category. On page 251 they say that 'an interesting problem is the construction of a nerve that is small like the Čech nerve and rigid like the Vietoris nerve'.

The modified Čech homology is one possible answer to this question. After all, the nerves of the underlying open coverings are small. Another somewhat thriftier way to rigidify the Čech construction is to mimic the construction of Chogoshvili ([20]). This was done in [51] after Edwards and Hastings: the Vietoris nerve is again replaced by the Čech nerve, but the category of coverings is arranged to be left filtering as follows. For a compact Hausdorff space X, let $\mathcal{A}$ be the set of all finite decompositions of the set X. An element $\mathcal{E} = (E_1, \ldots, E_k) \in \mathcal{A}$ consists of arbitrary subsets $E_i \subset X$ such that $E_i \cap E_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{k} = X$. $\mathcal{A}$ is ordered by inclusions. Let $\mathbf{Cov}(X)$ be the set of pairs $(\mathcal{E},\mathcal{U})$, where $\mathcal{E} = \{E_i\}$ is a decomposition from $\mathcal{A}, \mathcal{U} = \{U_i\}$ is a finite open covering of X with $\overline{E_i} \subset U_i$ for all i so that this correspondence indices an isomorphism of the nerves $N\overline{\mathcal{E}} \cong N\mathcal{U}$. For two elements $\Sigma = (\mathcal{E},\mathcal{U})$, $\Delta = (\mathcal{D},\mathcal{V})$, say that $\Sigma > \Delta$ if $\mathcal{E} > \mathcal{D}$ and $U_j \subset V_i$ when $E_j \subset D_i$. With this ordering, $\mathbf{Cov}(X)$ is left filtering. The projection to the second coordinate gives a system cofinal in $\mathrm{Cov}^{\mathrm{s}}\, X$, and the nerves are defined by $N(\mathcal{E},\mathcal{U}) = N\mathcal{U}$.

2.3. COFINALITY IN ČECH THEORY

First, we define some operations in $Cov\, X$.

(1) Given $\beta \in Cov\, X$, define $\cap\beta \in Cov\, X$ by

$$(\cap\beta)(x) = \bigcap_{x\in\beta(z)} \beta(z).$$

Notice that $\cap\beta > \beta$. Another way to view this construction is as a canonical rigidification of the classical finite open covering $\mathrm{im}\,\beta$.

(2) Given a finite subset $\{\beta_i\} \subset Cov\, X$, define $\cap\beta_i \in Cov\, X$ by

$$(\cap\beta_i)(x) = \bigcap_{i,x\in\beta_i(z)} \beta_i(z).$$

Notice that $\cap\beta_i > \beta_i$ for every index i, and $\cap\beta = \beta \cap \beta$.

(3) Given a finite subset $\{\beta_i\} \subset Cov\, X$, define $\times\beta_i \in Cov\, X$ by

$$(\times\beta_i)(x) = \bigcap_{i} \beta_i(x).$$

Again, $\times\beta_i > \beta_j$ for every index j.

Choose and fix a (left filtering) subcategory $\imath\colon \mathcal{C} \hookrightarrow Cov\, X$ closed under the $\times$-operation.

It is necessary to enlarge morphism sets in $Cov\, X$. Let $Cov^{\mathrm{is}}\, X$ be the category of set maps $\beta\colon X \to \mathcal{O}(X)$, the open subsets of X, such that $\mathrm{im}\,\beta$ is a finite set satisfying the two conditions from Definition 2.2.2. Morphisms $\phi\colon \beta_1 \to \beta_2 \in Cov^{\mathrm{is}}\, X$ are set endomorphisms $\phi\colon X \to X$ with the property that $\beta_1(x) \subset \beta_2(\phi(x))$ for all $x \in X$; they will be called *soft refinements*. The existence of such a refinement is denoted by $\beta_1 \succeq \beta_2$. If ϕ is realized by the identity map, we call ϕ a *rigid refinement*, denoted by $\beta_1 \geqslant \beta_2$. The subcategory of $Cov^{\mathrm{is}}\, X$ with only rigid morphisms is precisely $Cov\, X$.

For each morphism $\phi \in Cov^{\mathrm{is}}\, X$, let $\ominus\phi$ denote the domain and $\odot\phi$ the range of $\phi\colon \ominus\phi \to \odot\phi$. Consider the subcategory $\mathcal{M}'$ of the category of morphisms of

$Cov^{\text{is}}\, X$ such that $\phi \in \mathcal{M}'$ iff (1) $\ominus\phi \in \mathcal{C}$, and (2) $\mathrm{im}\,(\odot\phi) \circ \phi = \mathrm{im}\,(\odot\phi)$ and $\mu\colon \phi_1 \to \phi_2 \in \mathcal{M}'$ iff (1) $\ominus\mu\colon \ominus\phi_1 \to \ominus\phi_2 \in \mathcal{C}$ and (2) $\odot\mu\colon \odot\phi_1 \to \odot\phi_2 \in Cov\, X$. Notice that $\mu\colon \phi_1 \to \phi_2 \in \mathcal{M}'$ forces $\phi_1 = \phi_2$ as set maps. This comes from the requirement that (‡) $\phi_2 \circ \ominus\mu = \odot\mu \circ \phi_1$. The same requirement implies that (3) $\odot\phi_1(\phi_1(x)) \subset \odot\phi_2(\phi_2(x))$ which becomes simply the realization of $\odot\mu$. Let us form a new category $\mathcal{M}$ with same elements as in $\mathcal{M}'$ and morphisms $\mu\colon \phi_1 \to \phi_2$ being pairs $(\ominus\mu, \odot\mu)$ satisfying (1), (2), and (3). The essence is that the weaker property (3) replaces (‡) from $\mathcal{M}'$. Consider also the subcategory $\mathcal{P}$ of the category of morphisms of $Cov\, X$ with $\phi \in \mathcal{P}$ iff $\ominus\phi \in \mathcal{C}$. It can be viewed as a subcategory of $\mathcal{M}$ with the inclusion denoted by $i\colon \mathcal{P} \to \mathcal{M}$.

DEFINITION 2.3.1. Let $\Theta\colon \mathcal{M} \to \mathcal{P}$ be the functor determined by

– $\ominus\Theta(\mu) = \ominus\mu$,
– $\odot\Theta(\mu)(x) = \odot\mu(\mu(x))$.

Functoriality of the construction follows from property (3) of morphisms in $\mathcal{M}$.

There are two functorial projections, $\pi_1\colon \mathcal{M} \to \mathcal{C}$, $\mu \mapsto \ominus\mu$, and $\pi_2\colon \mathcal{M} \to Cov\, X$, $\mu \mapsto \odot\mu$. The same notation will be used for similar $\pi_1\colon \mathcal{P} \to \mathcal{C}$, $\pi_2\colon \mathcal{P} \to Cov\, X$.

LEMMA 2.3.2. *The induced map of homotopy limits*

$$\imath^*\colon \underset{\overleftarrow{\pi_2(\mathcal{P})}}{\mathrm{holim}}(N\_ \wedge KR) \to \underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}(N\_ \wedge KR)$$

is a weak homotopy equivalence.

Proof. Since $\imath\colon \mathcal{C} \hookrightarrow \pi_2(\mathcal{P})$ is cofinal in the classical sense, the Modified Cofinality Lemma 1.1.6 applies.

PROPOSITION 2.3.3. *Suppose $\pi_2\colon \mathcal{M} \to Cov\, X$ is an epimorphism. In particular, the inclusion $i = \ell \circ \jmath\colon \mathcal{C} \hookrightarrow Cov^{\text{is}}\, X$ is cofinal. Then the induced map of homotopy limits*

$$\jmath^*\colon \check{h}(X; KR) = \underset{\overleftarrow{Cov\, X}}{\mathrm{holim}}(N\_ \wedge KR) \to \underset{\overleftarrow{\mathcal{C}}}{\mathrm{holim}}(N\_ \wedge KR)$$

is a weak homotopy equivalence.

Proof. Since $\mathcal{P}$ is a partially ordered set, we will interpret its elements as pairs: $(\sigma, \delta) \in \mathcal{P} \subset \mathcal{C} \times Cov\, X$ iff $\sigma \in \mathcal{C}$, $\sigma \geqslant \delta$. Consider the functor $\Phi\colon \mathcal{P} \to \mathcal{P}$ given by $(\sigma, \delta) \mapsto (\sigma, \sigma)$. We start by checking that Φ is left cofinal. So let $\sigma \in \mathcal{C}$ and suppose (σ_1, δ_1), $(\sigma_2, \delta_2) \in \mathcal{P}$ with $\sigma_1 \geqslant \sigma$, $\sigma_2 \geqslant \sigma$. To prove that $\Phi \downarrow \sigma$ is left filtered, we need to exhibit $(\sigma_3, \delta_3) \in \mathcal{P}$ with $\sigma_3 \geqslant \sigma$ and $(\sigma_3, \delta_3) \geqslant (\sigma_i, \delta_i)$, $i = 1, 2$. Our choice is $(\sigma_1 \times \sigma_2, \delta_1 \times \delta_2)$. (Notice that it is not always true that $\sigma_1 \cap \sigma_2 \geqslant \delta_1 \cap \delta_2$.) The existence of equalizers in $\Phi \downarrow \sigma$ is obvious, so $\Phi \downarrow \sigma$ is left filtering, hence contractible. The over categories are nonempty since $(\sigma, \sigma) \geqslant (\sigma, \delta) \in \mathcal{P}$.

We would like to claim that $\pi_1: \mathcal{M} \to \mathcal{C}$ and $\pi_2: \mathcal{M} \to \mathit{Cov}\, X$ are both left cofinal. The hypothesis makes every over category associated to π_2 nonempty. It is also clear that each $\pi_2{\downarrow}\delta$, $\delta \in \mathit{Cov}\, X$, has equalizers. It will be contractible if $\mathcal{M}$ is shown to be left filtered.

Given $\phi_i: \sigma_i \to \delta_i \in \mathcal{M}$, $i = 1, 2$, consider $\delta_1 \cap \delta_2 \in \mathit{Cov}\, X$. When X is connected, $(\delta_1 \cap \delta_2)^{-1}(U)$ is uncountable for $U \in \operatorname{im}(\delta_1 \cap \delta_2)$, so there is $\phi_{\text{aux}}: \sigma_{\text{aux}} \to \delta_1 \cap \delta_2 \in \mathcal{M}$, where ϕ_{aux} is a set automorphism of X. Since $x \in (\sigma_1 \times \sigma_2)(x) \cap (\delta_1 \cap \delta_2)(\phi_{\text{aux}}(x))$, ϕ_{aux} can be chosen with the property $\phi_{\text{aux}}(x) \in (\sigma_1 \times \sigma_2)(x)$. This gives $x \in (\sigma_1 \times \sigma_2) \circ \phi_{\text{aux}}^{-1}(x)$. Construct new coverings $\sigma_i' = \sigma_i \circ \phi_{\text{aux}}^{-1} \in \mathit{Cov}^{\text{is}}\, X$, $i = 1, 2$. Let $\sigma = \sigma_1 \times \sigma_2 \times \sigma_{\text{aux}}$, $\delta = \sigma_1' \times \sigma_2' \times (\delta_1 \cap \delta_2)$, and define $\phi: \sigma \to \delta$ to be ϕ_{aux}. Now we check: (1) $\sigma(x) \subset (\sigma_1 \times \sigma_2)(x) = (\sigma_1' \times \sigma_2')(\phi_{\text{aux}}(x))$, $\sigma(x) \subset \sigma_{\text{aux}}(x) \subset (\delta_1 \cap \delta_2)(\phi_{\text{aux}}(x))$. So $\phi: \sigma \succeq \delta$. (2) $\sigma \geqslant \sigma_1 \times \sigma_2 \geqslant \sigma_i$, $\delta \geqslant \delta_1 \cap \delta_2 \geqslant \delta_i$ for $i = 1, 2$. (3) $\delta(\phi(x)) \subset (\sigma_1' \times \sigma_2')(\phi(x)) = (\sigma_1 \times \sigma_2)(x) \subset \sigma_i(x) \subset \delta_i(\phi_i(x))$ for $i = 1, 2$. We get the desired morphisms $\phi \to \phi_1$, $\phi \to \phi_2$.

The projection $\pi_1: \mathcal{M} \to \mathcal{C}$ is also an epimorphism on objects. Similar reasoning shows that each over category of π_1 is also left filtering. In fact, it has even simpler structure: the pair $(\sigma \geqslant \mathcal{X}, \sigma \geqslant \sigma)$, where $\mathcal{X}: x \mapsto X$ for each $x \in X$, is the terminal object in $\pi_1{\downarrow}\sigma$.

Now π_1 and π_2 are left cofinal functors. It follows also from the left filtering property of $\mathcal{M}$ and the very functoriality of the Θ-construction that $\Theta: \mathcal{M} \to \mathcal{P}$ is likewise left cofinal.

Now the map $\jmath^*$ can be embedded as the bottom row of the following commutative diagram:

$$\begin{tikzcd}
\underleftarrow{\operatorname{holim}}_{\mathcal{M}}(N\pi_2_ \wedge KR) \arrow[r] & \underleftarrow{\operatorname{holim}}_{\mathcal{M}}(N\pi_1_ \wedge KR) \\
\underleftarrow{\operatorname{holim}}_{\mathit{Cov}\, X}(N_ \wedge KR) \arrow[u, "\pi_2^*"] \arrow[r, "\jmath^*"] & \underleftarrow{\operatorname{holim}}_{\mathcal{C}}(N_ \wedge KR) \arrow[u, "\pi_1^*"]
\end{tikzcd}$$

The vertical arrows are weak homotopy equivalences by the Cofinality Lemma. The top arrow can be interpreted as follows. If $\Psi: \mathcal{M} \to \mathcal{M}$ is the projection analogous to Φ, restricting to Φ, notice that $N\pi_1\_ \wedge KR = N\pi_2\Psi\_ \wedge KR$. So, if denote $N\pi_2\_ \wedge KR$ by $G(\_)$, the top arrow is clearly

$$\Psi^*: \underleftarrow{\operatorname{holim}}_{\mathcal{M}} G \longrightarrow \underleftarrow{\operatorname{holim}}_{\mathcal{M}}(G \circ \Psi)$$

from the commutative square $\pi_2 \circ \Psi = \imath \circ \pi_1$.

Consider the commutative diagram

$$\begin{array}{ccccc}
\underset{\overleftarrow{\mathcal{P}}}{\operatorname{holim}}(N\pi_2 i_\wedge KR) & \xrightarrow{\Phi^*} & \underset{\overleftarrow{\mathcal{P}}}{\operatorname{holim}}(N\pi_2 i\Phi_\wedge KR) & = & \underset{\overleftarrow{\mathcal{P}}}{\operatorname{holim}}(N\pi_2 \Psi i_\wedge KR) \\
\uparrow{\scriptstyle i^*} & & & & \uparrow{\scriptstyle i^*} \\
\underset{\overleftarrow{\mathcal{M}}}{\operatorname{holim}}(N\pi_2_\wedge KR) & & \xrightarrow{\qquad\Psi^*\qquad} & & \underset{\overleftarrow{\mathcal{M}}}{\operatorname{holim}}(N\pi_2 \Psi_\wedge KR)
\end{array}$$

Since Φ is left cofinal, we are done as soon as both inclusion induced maps are shown to be weak equivalences. Notice that in the natural transformation $T\colon N\pi_2\Theta \to N\pi_2$ induced by the vertex maps $\odot\Theta\phi(x) \mapsto \odot\phi(\phi(x))$ all of $T\phi$, $\phi \in \mathcal{M}$, are homotopy equivalences by Quillen's Theorem A. Since $\Theta \circ i = \mathrm{id}$, we get the commutative diagram

$$\begin{array}{ccc}
\underset{\overleftarrow{\mathcal{M}}}{\operatorname{holim}}(N\pi_2\Theta_\wedge KR) & \xrightarrow{T_*} & \underset{\overleftarrow{\mathcal{M}}}{\operatorname{holim}}(N\pi_2_\wedge KR) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
\underset{\overleftarrow{\mathcal{P}}}{\operatorname{holim}}(N\pi_2\Theta i_\wedge KR) & = & \underset{\overleftarrow{\mathcal{P}}}{\operatorname{holim}}(N\pi_2 i_\wedge KR)
\end{array}$$

The vertical map on the left is the left inverse of Θ^* which is a weak equivalence. This i^* and T_* being weak equivalences proves that i^* on the right is one, too. Similarly, the other map i^* in the previous diagram is a weak homotopy equivalence.

Remark 2.3.4. When the inclusion i of $\mathcal{C}$ in $Cov^{\mathrm{is}}\,X$ is left cofinal, there is a reason to expect that $\jmath^*$ is again a weak homotopy equivalence. Just as in the case of cofinal i, the evidence comes from the Bousfield–Kan spectral sequence (Theorem 1.1.5), since the induced homomorphisms

$$\jmath^*\colon \underset{\overleftarrow{Cov\,X}}{\lim{}^{\mathrm{p}}}\ \pi_q(N\_\wedge KR) \longrightarrow \underset{\overleftarrow{\mathcal{C}}}{\lim{}^{\mathrm{p}}}\ \pi_q(N\_\wedge KR)$$

of the entries in the E_2-terms coincide with

$$(F'\circ\jmath)^*\colon \underset{\overleftarrow{\mathrm{Cov}\,X}}{\lim{}^{\mathrm{p}}}\ \pi_q(N\_\wedge KR) \longrightarrow \underset{\overleftarrow{F'(\mathcal{C})}}{\lim{}^{\mathrm{p}}}\ \pi_q(N\_\wedge KR).$$

Here $F'\colon Cov\,X \to \mathrm{Cov}\,X$ is the obvious extension of $F\colon Cov\,X \to Cov^{\mathrm{s}}X$ (see 2.2). It follows from the homotopy theoretic interpretation of derived limits ([10, XI, 7.2]) and the weak equivalence

$$(F'\circ\jmath)^*\colon \underset{\overleftarrow{\mathrm{Cov}\,X}}{\operatorname{holim}} K(\pi_q(N\_\wedge KR),n) \longrightarrow \underset{\overleftarrow{F'(\mathcal{C})}}{\operatorname{holim}} K(\pi_q(N\_\wedge KR),n)$$

that all $\jmath^*_{p,q}$ are isomorphisms.

3. The Approach to Novikov Conjectures

This section restates Theorem 2 from the Introduction in a more precise form. In particular, the map is defined between the homotopy limits in the statement which is expected to be a weak homotopy equivalence. Thus Sections 3.1–3.3 are a sketch of the approach of Carlsson and Pedersen to Novikov conjectures. Then we describe the version of this approach which we actually use in this paper.

3.1. CONTINUOUS CONTROL AT INFINITY

First, we copy some definitions from [16]. Let E be a topological space, and let $R[E]^\infty$ denote the free R-module generated by $E \times \mathbb{N}$. The category $\mathcal{B}(E;R)$ is defined to consist of submodules A of $R[E]^\infty$ such that denoting $A \cap R[x]^\infty, x \in E$, by A_x we have $A = \oplus A_x$, each A_x is a finitely generated free R-module, and $\{x : A_x \neq 0\}$ is locally finite in E. Morphisms are all R-module homomorphisms. Note that a Γ-action on X always induces a Γ-action on $\mathcal{B}(E;R)$. Also $\mathcal{B}(E;R)$ is a small additive category.

If X is a topological space, Y a subspace, $E = X - Y$, $U \subset X$ is any subset, and $A \in \mathcal{B}(E;R)$, define $A|U$ by $(A|U)_x = A_x$ if $x \in U - Y$ and 0 if $x \in X - U - Y$. A morphism $\phi\colon A \to B$ in $\mathcal{B}(E;R)$ is called *continuously controlled* at $y \in Y$ if for every neighborhood U of y in X there is a neighborhood V so that $\phi(A|V) \subset B|U$ and $\phi(A|X - U) \subset B|X - V$.

Now let T be an open subset of X and $p\colon T \to K$ be a map with continuous $p|Y \cap T$. A morphism $\phi\colon A \to B \in \mathcal{B}(E;R)$ is *p-controlled* at $y \in Y \cap T$ if for every neighborhood U of $p(y)$ in K there is a neighborhood V of y in X so that $\phi(A|V) \subset B|p^{-1}(U)$ and $\phi(A|X - p^{-1}(U)) \subset B|X - V$.

The category $\mathcal{B}(X,Y;R)$ has the same objects as $\mathcal{B}(E;R)$ and morphisms which are continuously controlled at all $y \in Y$. The category $\mathcal{B}(X,Y,p;R)$ has the same objects as $\mathcal{B}(E;R)$ and morphisms which are continuously controlled at all $y \in Y - T$ and p-controlled at all $y \in T \cap Y$. These are small symmetric monoidal categories, so there are corresponding nonconnective K-theory spectra defined as in Section 1.2. We will use the notation $K(\_)$ for $K(\mathcal{B}(\_))$.

3.2. PROOF OF THEOREM 1

Here is the general scheme of the approach used in [16] to prove Theorem 1 from the Introduction. Let $C\widehat{X}$ be the cone on $\widehat{X}$ with $\widehat{X} = \widehat{X} \times \{1\}$, $Y = \widehat{X} - E\Gamma$, and $p\colon \widehat{X} \times (0,1) \to \widehat{X}$ be the projection. The map $\pi\colon C\widehat{X} \to \Sigma\widehat{X}$ collapsing $\widehat{X}$ induces a Γ-equivariant functor

$$\mathcal{B}(C\widehat{X}, CY \cup \widehat{X}, p; R) \xrightarrow{\pi_*} \mathcal{B}(\Sigma\widehat{X}, \Sigma Y, p; R)$$

which in its turn induces a map of spectra

$$\mathcal{S} = \Omega K(C\widehat{X}, CY \cup \widehat{X}, p; R) \xrightarrow{\pi_*} \mathcal{T} = \Omega K(\Sigma\widehat{X}, \Sigma Y, p; R).$$

Next they show that there is a commutative diagram

$$\begin{array}{ccc} B\Gamma_+ \wedge KR & \xrightarrow{\alpha} & K(R\Gamma) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathcal{S}^\Gamma & \xrightarrow{\pi_*^\Gamma} & \mathcal{T}^\Gamma \end{array}$$

Recall that the fixed point spectrum of a Γ-spectrum $\mathcal{A}$ can be defined as $\mathcal{A}^\Gamma = \mathrm{Map}_\Gamma(S^0, \mathcal{A}_+)$. The homotopy fixed point spectrum can be defined analogously: $\mathcal{A}^{h\Gamma} = \mathrm{Map}_\Gamma(E\Gamma_+, \mathcal{A}_+)$. The collapse $\rho\colon E\Gamma_+ \to S^0$ induces $\rho^*\colon \mathcal{A}^\Gamma \to \mathcal{A}^{h\Gamma}$. Such maps make the next diagram commute:

$$\begin{array}{ccc} \mathcal{S}^\Gamma & \xrightarrow{\pi_*^\Gamma} & \mathcal{T}^\Gamma \\ \downarrow \rho^* & & \downarrow \\ \mathcal{S}^{h\Gamma} & \xrightarrow{\pi_*^{h\Gamma}} & \mathcal{T}^{h\Gamma} \end{array}$$

It is shown in [16] that $\rho^*\colon \mathcal{S}^\Gamma \simeq \mathcal{S}^{h\Gamma}$ and $\pi_*^{h\Gamma}\colon \mathcal{S}^{h\Gamma} \simeq \mathcal{T}^{h\Gamma}$. Putting the two diagrams together we see that this is enough to make α a split injection. Note that very little is known about the map $\mathcal{T}^\Gamma \to \mathcal{T}^{h\Gamma}$, but only being a part of the commutative diagram is required of it.

3.3. PROOF OF THEOREM 2

Now let us consider the circumstances of Theorem 2 following [17]. Consider another map κ with domain $C\widehat{X}$ which contracts the subspace CY and produces the reduced cone $\widetilde{C}E\Gamma^+$. It induces a Γ-equivariant functor

$$\mathcal{B}(C\widehat{X}, CY \cup \widehat{X}, p; R) \xrightarrow{\kappa_*} \mathcal{B}(\widetilde{C}E\Gamma^+, E\Gamma^+; R).$$

Notice that each morphism from $\mathcal{B}(C\widehat{X}, CY \cup \widehat{X}, p; R) \subset \mathcal{B}(X \times \mathbb{R})$ is controlled at $E\Gamma^+$. This functor induces a map of spectra

$$\mathcal{S} = \Omega K(C\widehat{X}, CY \cup \widehat{X}, p; R) \xrightarrow{\kappa_*} \mathcal{R} = \Omega K(\widetilde{C}E\Gamma^+, E\Gamma^+; R).$$

PROPOSITION 3.3.1. *κ_* is a weak homotopy equivalence.*

$E\Gamma^+$ is metrizable, so, according to Theorem 1.36 of [16], $\mathcal{R}$ is a Steenrod functor, and $\mathcal{R}^\Gamma \simeq \mathcal{S}^\Gamma \longrightarrow \mathcal{T}^\Gamma$ is again the assembly map. Also $\mathcal{R}^\Gamma \simeq \mathcal{R}^{h\Gamma}$ as before. Another Steenrod functor is the Čech homology

$$\check{h}(E\Gamma^+; KR) = \underset{\mathcal{U} \in Cov\, E\Gamma^+}{\underleftarrow{\mathrm{holim}}} (N\mathcal{U} \wedge KR),$$

where $Cov\ E\Gamma^+$ is the category of finite rigid open coverings of $E\Gamma^+$. The nerve functor $N\colon Cov\ E\Gamma^+ \to$ S–SETS above lands in the category of simplicial sets. So $N\mathcal{U} \wedge KR$ above is a simplicial spectrum (see Section 2.2).

The *support at infinity* of an object $A \in \mathcal{B}(X, Y; R)$ is the set of limit points of $\{x : A_x \neq 0\}$ in Y. The full subcategory of $\mathcal{B}(X, Y; R)$ of objects with support at infinity contained in $C \subset Y$ is denoted by $\mathcal{B}(X, Y; R)_C$. If U_1, U_2 are open sets in $E\Gamma^+$ then we get maps induced by inclusions:

$$K(\widetilde{C}E\Gamma^+, E\Gamma^+; R)_{U_1 \cap U_2} \longrightarrow K(\widetilde{C}E\Gamma^+, E\Gamma^+; R)_{U_i}.$$

In general, there is a functor $Int\,\mathcal{U} \to$ SPECTRA for any $\mathcal{U} \in Cov\,E\Gamma^+$, where $Int\,\mathcal{U}$ is the partially ordered set of all multiple intersections of members of $\mathcal{U}$ (indexed by finite subsets of Y).

PROPOSITION 3.3.2. *For a fixed $\mathcal{U} \in Cov\,E\Gamma^+$ the universal excision map*

$$\underset{\overrightarrow{Int\,\mathcal{U}}}{\operatorname{hocolim}} K(\widetilde{C}E\Gamma^+, E\Gamma^+; R)_{\cap U_i} \longrightarrow K(\widetilde{C}E\Gamma^+, E\Gamma^+; R)$$

is a weak equivalence.

The spectrum $\Sigma\mathcal{R}$ on the right is a Γ-spectrum. To rediscover this aspect of the structure on the left, we can write

$$\underset{\overleftarrow{\mathcal{U} \in Cov\,E\Gamma^+}}{\operatorname{holim}} \left(\underset{\overrightarrow{Int\,\mathcal{U}}}{\operatorname{hocolim}} K(\widetilde{C}E\Gamma^+, E\Gamma^+; R)_{\cap U_i} \right) \simeq \Sigma\mathcal{R},$$

where the Γ-action on the left-hand side is induced from the obvious action on $Cov\,E\Gamma^+$. Notice that we have used the fact that $Cov\,E\Gamma^+$ is contractible in applying Theorem 1.1.4 to $\operatorname{holim}\Sigma\mathcal{R}$.

In the proper setup (essentially sending each nonempty $\cap U_i$ to a point) one gets maps

$$\underset{\overrightarrow{Int\,\mathcal{U}}}{\operatorname{hocolim}} K(\widetilde{C}E\Gamma^+, E\Gamma^+; R)_{\cap U_i} \longrightarrow |Int\,\mathcal{U}| \wedge KR.$$

Finally, we get the induced equivariant map of homotopy limits

$$\pi \colon \mathcal{R} \longrightarrow \check{h}(E\Gamma^+; KR).$$

This map can be viewed as a component of a natural transformation of Steenrod functors which is an equivalence on points, hence $\mathcal{R} \simeq \check{h}(E\Gamma^+; KR)$, according to Milnor (see [19, Lemma 3.3]. This is enough to conclude that $\mathcal{R}^{h\Gamma} \simeq \check{h}(E\Gamma^+; KR)^{h\Gamma}$.

Returning to $\mathcal{T}$, there is an excision result analogous to Proposition 3.3.2. In order to produce a natural transformation analogous to π, the covering sets $p(U) \subset Y$ must be boundedly saturated. See [17] for the construction of a map

$$\pi_\ell \colon \mathcal{T} \longrightarrow \Sigma \underset{\overleftarrow{\alpha \in \{\alpha\}}}{\operatorname{holim}} (N\alpha \wedge KR)$$

for each Γ-closed contractible system of coverings of Y by boundedly saturated open sets.

Again, this map is Γ-equivariant, so the composition induces a map

$$\mathcal{T}^{h\Gamma} \longrightarrow \left(\Sigma \operatorname*{holim}_{\overleftarrow{\alpha\in\{\alpha\}}} N\alpha \wedge KR \right)^{h\Gamma}.$$

Since $\widehat{X}$ is Čech-acyclic, there is a composite weak equivalence

$$\begin{array}{ccc} \operatorname*{holim}_{\overleftarrow{Cov(\widehat{X}\cup CY)}} (N\_ \wedge KR) & \xrightarrow{\simeq} & \operatorname*{holim}_{\overleftarrow{Cov\,\Sigma Y}} (N\_ \wedge KR) \\ \big\downarrow\simeq & & \big\uparrow\simeq \\ \operatorname*{holim}_{\overleftarrow{Cov\,E\Gamma^+}} (N\_ \wedge KR) & \xrightarrow{\exists} & \Sigma\operatorname*{holim}_{\overleftarrow{Cov\,Y}} (N\_ \wedge KR) \end{array}$$

There is a map

$$\theta\colon \operatorname*{holim}_{\overleftarrow{Cov\,Y}} (N\_ \wedge KR) \longrightarrow \operatorname*{holim}_{\overleftarrow{\{\alpha\}}} (N\_ \wedge KR)$$

induced by the inclusion of categories $\{\alpha\} \hookrightarrow Cov\,Y$; it makes the ambient diagram commutative. If θ is a weak equivalence then $\theta^{h\Gamma}$ is a weak equivalence. This would make α a split injection, since a weak equivalence would again be factored as the composition of α with another map.

Summary 3.3.3. Given a discrete group Γ, the method described here calls for a construction of a compact classifying space $B\Gamma$ and an equivariant compactification $\widehat{X}$ of the universal cover $E\Gamma$, i.e., an open dense embedding $E\Gamma \hookrightarrow \widehat{X}$ in a compact Hausdorff space. The space $\widehat{X}$ itself may not be metrizable but it is required to be acyclic in the sense that its Čech homology is that of a point. Then a convenient metric must be introduced on $E\Gamma$. The action may not be small at infinity, but the choice of a metric determines the family of boundedly saturated subsets of $Y = \widehat{X} - E\Gamma$. One has to make a choice of a Γ-invariant collection of coverings of Y by such sets which preserves the Čech homotopy type of Y. Furthermore, the weak homotopy equivalence of Čech homology spectra has to be realized by the map θ defined above.

3.4. MODIFICATION

The flexibility of this approach is in the freedom of choice of the metric in $\bar{X}$ and the system of special coverings $\{\alpha\}$. It happens to be not enough for making θ a weak equivalence in a situation like ours when the choice of the metric is convenient and natural but makes the family of open boundedly saturated sets in $Y = \widehat{X} - \bar{X}$ too coarse to preserve the Čech homotopy type.

DEFINITION 3.4.1. Let C_1 and C_2 be two closed subsets of Y. The pair (C_1, C_2) is called *excisive* if there is an open subset $V \subset \widehat{X}$ such that $C_2 - C_1 \subset V$ and $\overline{V} \cap C_1 \subset C_2$. For two arbitrary subsets U_1 and U_2, the pair (U_1, U_2) is *excisive* if every compact subset C of $U_1 \cup U_2$ is contained in $C_1 \cup C_2$ where (C_1, C_2) is an excisive pair of closed subsets with $C_i \subset U_i$. A collection of subsets $U_i \subset Y$ is called *excisive* if every pair in the Boolean algebra of sets generated by U_i is excisive.

It is easy to show that $\operatorname{Cov} X$ for compact Hausdorff X consists of excisive coverings. It turns out that this property is sufficient for the excision result like Proposition 3.3.2 (see the proof in [17]). Our choice for $\{\alpha\}$ will be certain excisive coverings by boundedly saturated sets so that the category itself is contractible. This makes possible the construction of a map similar to π_ℓ above. Since $\Sigma\check{h}(Y; KR)$ is weakly equivalent to the domain of $\pi_*\colon \mathcal{S} \to \mathcal{T}$ such that $(\pi_*)^\Gamma$ is the assembly map, there must be a map

$$\theta\colon \check{h}(Y; KR) = \underset{\overleftarrow{Cov\, Y}}{\operatorname{holim}}(N\_ \wedge KR) \longrightarrow \underset{\overleftarrow{\{\alpha\}}}{\operatorname{holim}}(N\_ \wedge KR)$$

which completes the commutative diagram.

To create a natural target for a map from $\mathcal{T}$ we can ‘saturate’ the open sets $U \subset Y$ by associating to U its envelope in a Boolean algebra of boundedly saturated subsets of Y thus mapping $Cov\, Y$ functorially onto the resulting category $\{\alpha\}$. Let us denote this functor by $\mathsf{sat}\colon \beta \mapsto \alpha(\beta)$. Since sat is left cofinal, and the construction $\beta \rightsquigarrow \alpha(\beta)$ above induces a natural transformation of the functors $N\beta \wedge KR \to N\alpha(\beta) \wedge KR$ from $Cov\, Y$ to S-SPECTRA, we can induce and compose the following maps:

$$\theta\colon \underset{\overleftarrow{\beta \in Cov\, Y}}{\operatorname{holim}} (N\beta \wedge KR) \xrightarrow{\mathsf{sat}_*} \underset{\overleftarrow{\beta \in Cov\, Y}}{\operatorname{holim}} N\alpha(\beta) \wedge KR \xleftarrow{\simeq} \underset{\overleftarrow{\alpha \in \{\alpha\}}}{\operatorname{holim}}(N\alpha \wedge KR).$$

This is the correct map if we make sure that the analogue of the excision result from [17] works with $\{\alpha\}$. It is precisely the property of $A \in \alpha$ being excisive that we need here. This cannot be always guaranteed. However, one can often make a more intelligent choice of the Boolean subalgebra of boundedly saturated sets in the construction of $\{\alpha\}$. Taking envelopes in this algebra defines all the analogues of the maps above with all the same properties.

Now $\{\alpha\}$ may not be included in $Cov\, Y$ any longer. This is why one is forced to consider the more general situation. We will pass to a convenient intermediate category $\mathcal{C}$ of $Cov\, Y$ where the open covering sets have particularly nice nature so that it is easy to predict the saturation and see that it does not change the homotopy type of the nerve. The passage is achieved using the following diagram.

$$\begin{array}{ccccc}
\underset{\substack{\longleftarrow \\ \beta\in Cov\,Y}}{\text{holim}}\ (N\beta\wedge KR) & \xrightarrow{\mathsf{sat}_*} & \underset{\substack{\longleftarrow \\ \beta\in Cov\,Y}}{\text{holim}}\ N\alpha(\beta)\wedge KR & \xleftarrow{\simeq} & \underset{\substack{\longleftarrow \\ \alpha\in\{\alpha\}}}{\text{holim}}(N\alpha\wedge KR) \\
\big\downarrow{\scriptstyle \imath^*} & & \big\downarrow & & \big\downarrow \\
\underset{\substack{\longleftarrow \\ \mathcal{C}}}{\text{holim}}(N_\wedge KR) & \xrightarrow{(\mathsf{sat}|\mathcal{C})_*} & \underset{\substack{\longleftarrow \\ \mathcal{C}}}{\text{holim}} N\alpha(_)\wedge KR & \xleftarrow{\simeq} & \underset{\substack{\longleftarrow \\ \{\alpha'\}}}{\text{holim}}(N_\wedge KR).
\end{array}$$

The vertical maps are induced by inclusions. Now the map $\mathcal{T}\to$ holim$(N\alpha\wedge KR)$ can be composed with the vertical map on the right, so in order to split the assembly map we need $\imath^*$ and $(\mathsf{sat}|\mathcal{C})_*$ to be weak equivalences.

EXAMPLE 3.4.2. If there is a Γ-closed contractible category $\mathcal{D}$ of finite rigid open coverings by boundedly saturated sets then it can be taken to play the role of $\mathcal{C}$. In this case sat is an identity, so only $\imath^*$ needs to be an equivalence, and we recover Theorem 2 of Carlsson and Pedersen.

Our own choice of $\mathcal{C}$ will be explained in Section 9.1.

4. Malcev Spaces and their Compactification

We start our inductive constructions with a study of simply connected nilpotent groups. It could culminate in a proof of the Novikov conjecture for the class NIL of torsion-free finitely generated nilpotent groups. It is possible, however, to deal with these groups using different approaches via reduction ([15, 29, 48]). We actually compactify a suitable $E\Gamma$, and it is this construction that we are really after. We also use this format to organize some information needed later.

4.1. MALCEV COORDINATES

Let G be a real Lie group, and $\mathfrak{g}$ be its Lie algebra. There are several ways to introduce a local coordinate system in a neighborhood of the identity $e\in G$. If $\{X_1,\dots,X_d\}$ is a basis in $\mathfrak{g}$, introduce a coordinate system $\{u_1,\dots,u_d\}$ in $\mathfrak{g}$ by mapping

$$X=\sum_{i=1}^{d}u_iX_i\mapsto(u_1,\dots,u_d)\in\mathbb{R}^d.$$

For the usual norm $|X| = (\sum_{i=1}^{d}|u_i|^2)^{1/2}$ in $\mathfrak{g}$, there exists a number $\epsilon>0$ such that the exponential maps an open norm-metric ball at 0 in $\mathfrak{g}$ injectively and regularly into G. The image $U_e = \{\exp X : |X| < \epsilon\}$ is a neighborhood of e. If we take

$$x_k\left(\exp\sum_{i=1}^{d} u_i X_i\right) = u_k$$

then $\{x_1, \dots, x_d\}$ is a local *canonical coordinate system of the first kind*. Recall that if G is a connected simply connected nilpotent group, the exponential map is a global diffeomorphism, so the coordinate system $\{x_i\}$ is also global.

A *lattice* Γ in a connected Lie group G is a discrete subgroup such that G/Γ has finite volume. Let us begin with

THEOREM 4.1.1 (Malcev [42]). *A group is isomorphic to a lattice in a simply connected nilpotent Lie group if and only if it is finitely generated, nilpotent, and torsion-free.*

Let Γ be a torsion-free finitely generated nilpotent group which we embed in a connected simply connected group N produced by Theorem 4.1.1. This N will be the model for $E\Gamma$. By Lemma 4 of [42] the subgroup Γ has generators $\{\gamma_1, \dots, \gamma_r\}$, where $r = \dim N$, with the three properties:

1. each $\gamma \in \Gamma$ can be written as $\gamma = \gamma_1^{n_1} \cdots \gamma_r^{n_r}$,
2. each subset $\Gamma_i = \{\gamma_i^{n_i} \cdots \gamma_r^{n_r}\}$ is a normal subgroup of Γ, and
3. the quotients Γ_i/Γ_{i+1} are infinite cyclic for all $1 \leqslant i < r$.

Let $C_i = c_i(t)$ be the one-parameter subgroup of N with $c_i(1) = \gamma_i$, $1 \leqslant i \leqslant r$. It is easily seen that N satisfies analogues of the three properties of Γ:

1. $N = C_1 \cdots C_r$, and the representation of $g \in N$ as $g = g_1 \cdots g_r$, $g_i \in C_i$, is unique,
2. if $N_{r+1} = \{e\}$, $N_i = C_i \cdots C_r$, $1 \leqslant i \leqslant r$, then N_i are Lie subgroups of N, $\dim N_i = r - i + 1$, and $N_i \lhd N$ for $1 \leqslant i < r$,
3. $C_i \cong \mathbb{R}$ for all $1 \leqslant i \leqslant r$.

If n is the Lie algebra of N then $e_1 = \log\gamma_1, \dots, e_r = \log\gamma_r$ becomes a basis in n so that each set

$$\mathsf{n}_i = \{\alpha_i e_i + \alpha_{i+1} e_{i+1} + \cdots + \alpha_r e_r\} \subset \mathsf{n}$$

is an ideal. So $\{\gamma_i\}$ produce special *canonical coordinates of the first kind according to Malcev*. This system should not be confused with the canonical coordinate system of *the second kind* (or *Malcev coordinates*); it seems that this terminology first appeared in A. I. Malcev's work on rigidity in nilpotent groups [42]. The correspondences

$$\log\colon g \mapsto \log g,$$

$$\sigma\colon \sum_{k=1}^{r} \alpha_k e_k \mapsto \sum_{k=1}^{r} \alpha_k (0, \dots, \overset{k}{\frown}{1} \dots, 0)$$

define diffeomorphisms between N, n and $\mathbb{R}^r$ and induce flat metrics in N and n from the standard Euclidean metric in $\mathbb{R}^r$.

4.2. COMPACTIFICATION

Let $M_i = N/N_i = C_1 \cdots C_{i-1}$. Since n_i is an ideal in n, for any $a \in N$ the Poisson bracket $[a, e_i] \in N_{i+1}$. Denote the coordinates of p, $g \in N$ by ξ_i, η_i respectively, then the coordinates $\zeta_i(t)$ of $p \cdot g$ satisfy

$$\zeta_i = \xi_i + \eta_i + q_i(\xi_1, \dots, \xi_{i-1}, \eta_1, \dots, \eta_{i-1}),$$

where q_i are polynomials determined by the Campbell–Hausdorff formula. This shows that if $p \in N_j$ then $\xi_1 = \ldots = \xi_{j-1} = 0$ and ζ_k, $k < j$, are independent of $\xi_j, \dots, \xi_r$. We can conclude that $p \cdot g$ lies in the hyperplane $(\zeta_1, \dots, \zeta_{j-1}, *, \dots, *)$ parallel to N_j. So N acts from the right on the set of hyperplanes parallel to N_j. Since the formulae are polynomial, the action is continuous. Similar arguments apply to the left action. One can consider the equivariant enlargement of N by the equivalence classes of rays in M_{j+1} parallel to C_j.

Perform this construction inductively for all $j = r, \dots, 2$. In order to visualize and parametrize the resulting compactification νN of N, it is helpful to embed N as $(-1, 1)^r \subset \mathbb{R}^r$ in the most obvious fashion so that the orders of the coordinates coincide and the parallelism relation is preserved. We want to consider a sequence of certain collapses. The collapses we have in mind are performed in the boundary of the cube I^r and its successive quotients. The first collapse contracts

$$\{(x_1, \dots, x_{r-1}, *) \in I^r : \exists 1 \leqslant i \leqslant r-1 \text{ with } x_i = \pm 1\} \longrightarrow \text{point.}$$

We give this point the projective coordinates $(x_1, \dots, x_{r-1}, \flat)$. The set

$$\{(x_1, \dots, x_{r-1}, \flat) : \exists 1 \leqslant i \leqslant r-1 \text{ with } x_i = \pm 1\}$$

is the boundary of I^{n-1}. Now we induct on the dimension of the cube. For example, the collapse at the m-th stage can be described as

$$\{(x_1, \dots, x_{r-m}, *, \flat, \dots, \flat) \in I^{r-m+1} : \exists 1 \leqslant i \leqslant r-m \text{ with } x_i = \pm 1\}$$
$$\longrightarrow (x_1, \dots, x_{r-m-1}, \flat, \dots, \flat).$$

The process stops after $r-1$ stages when the points $(\pm 1, \flat, \dots, \flat)$ do not get identified. The end result is a topological ball B^r with the CW-structure consisting of two cells of each dimension $0, 1, \dots, r-1$ and one r-dimensional cell and a continuous composition of collapses $\rho\colon I^r \to B^r$. Each lower dimensional cell is the quotient of the appropriate face in ∂I^r: if the face F was defined by $x_i = \pm 1$ then $\dim \rho(F) = i$.

This discussion proves

PROPOSITION 4.2.1. *The compactification νN is both left and right equivariant with respect to the multiplication actions of N on itself. The orbits of the two actions in $\tau N = \nu N - N$ coincide with the cells in the cellular decomposition of the boundary.*

Remark 4.2.2. Our motive for Proposition 4.2.1 is, of course, that now the restricted actions of any lattice Γ in N extend to τN. What makes this situation nontrivial and does not allow one to use the spherical or ideal compactification of the flat space N is the fact that the left and right Γ-actions on N *cannot be by isometries*. Indeed, Γ would then be a crystallographic group which by a theorem of Bieberbach would intersect the translation subgroup of $\mathrm{Isom}(\mathbb{R}^r)$ in a normal free Abelian subgroup of finite index. This would contradict the possible nontriviality of the semi-direct product structure on Γ (as in the Heisenberg groups, for example).

5. Bounded Saturation in the Boundary Sphere

5.1. GENERAL PROPERTIES OF BOUNDEDLY SATURATED SETS

For any subset K of a metric space (X, d) let $K[D]$ denote the set $\{x \in X : d(x, K) \leqslant D\}$ which we call the *D-neighborhood* of K.

DEFINITION 5.1.1. Given a metric space (X, d) embedded in a topological space $\widehat{X}$ as an open dense subset, a set $A \subset Y = \widehat{X} - X$ is *boundedly saturated* if for every closed subset C of $\widehat{X}$ with $C \cap Y \subset A$, the closure of each D-neighborhood of $C \backslash Y$ for $D \geqslant 0$ satisfies $\overline{(C\backslash Y)[D]} \cap Y \subset A$. Clearly, it is enough to consider only those C with $C \cap Y = \overline{C\backslash Y} \cap Y$.

Convention. All of the spaces we consider in this paper have the property that if x is a cluster point of some sequence $\{x_i\}$ then there is a subsequence $\{x_j\}$ so that x is the only cluster point of $\{x_j\}$. For example, this is satisfied by any metrizable space. That the spaces from Section **??** and Section A.2 satisfy this condition follows immediately from the definition of basic neighborhoods. When we say that a sequence $\{x_i\}$ *converges* to x and write $x = \lim\{x_i\}$, we understand that the original sequence has been replaced by a converging subsequence.

LEMMA 5.1.2. *Let S be a subset of Y which is not boundedly saturated. Then there exists a point $y \in Y\backslash S$ and a sequence $\{y_i\} \subset X$ converging to y so that $\overline{\{y_i\}[D]} \cap S \neq \emptyset$ for some $D > 0$.*

Proof. By the hypothesis there is a closed subset $K \subset \widehat{X}$ with $K \cap Y \subset S$ and $\overline{(K \cap X)[D]}\backslash S \neq \emptyset$ for some $D > 0$. Let $y \in \overline{(K \cap X)[D]}\backslash S$. Then there exists a sequence $\{y_i\} \subset X$ converging to y with $d(y_i, K \cap X) \leqslant D$. Consider $K \cap \{y_i\}[D]$; if this set is bounded then $\{y_i\}$ is contained in the bounded set $(K \cap \{y_i\}[D])[D] \subset X$ which would make $y \in X$. So there is a sequence $\{z_i\} \subset K \cap X$ with $z_i \in K \cap \{y_i\}[D]$ and $\lim_{i\to\infty}\{z_i\} \in K \cap Y \subset S$.

THEOREM 5.1.3. *A subset $S \subset Y$ is boundedly saturated if for any closed set $C \subset \widehat{X}$ with $C \cap S = \emptyset$ and any $D > 0$, $S \cap \overline{(C \cap X)[D]} = \emptyset$.*

Proof. Apply Lemma 5.1.2 to the contrapositive statement.

Notice that the hypothesis of Theorem 5.1.3 is precisely that the complement of S in Y is boundedly saturated. So we get

COROLLARY 5.1.4. *The collection of boundedly saturated subsets of Y is closed with respect to taking complements, finite intersections and unions. In other words, it is a Boolean algebra of sets.*

5.2. THE METRIC IN N

We must begin by identifying the metric in N with respect to which the bounded saturation property of sets in τN will be defined. It will be not the Euclidean metric used to construct the boundary but the left invariant Riemannian metric obtained by introducing a suitable inner product in $\mathfrak{n}$. In this situation the diameter of a chosen fundamental domain F is bounded by some number D as is also the diameter of any Γ-translate of the domain.

DEFINITION 5.2.1. Let (X_1, d_1) and (X_2, d_2) be metric spaces. A *quasi-isometry* is a (not necessarily continuous) map $f\colon X_1 \to X_2$ for which there exist constants λ, ϵ and C such that (1) for every $x_2 \in X_2$ there exists $x_1 \in X_1$ with $d_2(f(x_1), x_2) \leqslant C$, and (2) $\frac{1}{\lambda} d_1(x_1, y_1) - \epsilon \leqslant d_2(f(x_1), f(y_1)) \leq \lambda d_1(x_1, y_1) + \epsilon$ for all x_1, y_1 in X_1.

The crucial property of our metric is that the group Γ with the word metric is embedded quasi-isometrically when viewed as a subgroup of N. Our choice for F will be the parallelogram spanned by the basis $\{\gamma_i\}$.

5.3. BOUNDED SATURATION: THE SEARCH

We develop a systematic method of looking for boundedly saturated subsets of Y. Let Z be a left Γ-space with a Γ-invariant open dense complete locally compact (so that bounded closed sets are compact) metric subspace Z^0 on which Γ acts freely, cocompactly, properly discontinuously by isometries. Then according to Milnor for $x_0 \in Z^0$ the embedding $\epsilon\colon \gamma \mapsto \gamma \cdot x_0$ of Γ with the word metric into Z^0 is a quasi-isometry. In the course of the proof one constructs a compact subset $B \subset Z^0$ such that $\Gamma \cdot B = \bigcup_{\gamma \in \Gamma} \gamma B = Z^0$. Suppose that in addition there is a right Γ-action on Z which (1) leaves Z^0 invariant, (2) commutes with the left action: $(\gamma_1 \cdot z) \cdot \gamma_2 = \gamma_1 \cdot (z \cdot \gamma_2)$ for any $z \in Z^0$, and (3) restricts to the right translation action on $\epsilon(\Gamma)$, i.e., $(\gamma_2 \cdot x_0) \cdot \gamma_1 = (\gamma_2 \gamma_1) \cdot x_0$ for all $\gamma_1, \gamma_2 \in \Gamma$.

THEOREM 5.3.1. *Let L be a boundedly saturated subset of $Z - Z^0$. Then* (1) *there is a point $z \in L$ which is a limit of $\epsilon(\Gamma)$, and* (2) *the right orbit $z \cdot \Gamma \subset Z - Z^0$ is contained entirely in L.*

Proof. (1) Take an arbitrary $z' \in L$, and let $\{z_i\}$ be a sequence of points in Z^0 with $\lim_{i\to\infty}\{z_i\} = z'$. Let $B \subset Z^0$ be a ball of radius R centered at x_0 with the property that $\Gamma \cdot \operatorname{int} B = Z^0$. So the quasi-isometry constant $C \leqslant 2R$. Then $\{z_i\}[2R]$ contains all translates of B which contain some z_i, $i \geqslant 1$. A quasi-isometry inverse to ϵ can be constructed by sending $z \mapsto \gamma(z)$ if $z \in \gamma(z) \cdot B$. So $\{\gamma(z_i) \cdot x_0\} \subset \{z_i\}[2R]$. The sequence $\{\gamma(z_i)\}$ is unbounded, hence there is

$$z = \lim_{i\to\infty}\{\gamma(z_i) \cdot x_0\} \subset \overline{\{z_i\}[2R]} \cap (Z - Z^0) \subset L.$$

(2) Take the word metric k-ball B_k in Γ centered at e and act by it on $\{\gamma(z_i) \cdot x_0\}$ from the right. If $b \in B_k$ then $d(e,b) \leqslant k$, so $d(x_0, x_0 \cdot b) = d(x_0, b \cdot x_0) \leqslant \lambda k + \epsilon$, so $d(\gamma(z_i) \cdot x_0, (\gamma(z_i) \cdot x_0) \cdot b) = d(\gamma(z_i) \cdot x_0, \gamma(z_i) \cdot (x_0 \cdot b)) = d(x_0, x_0 \cdot b) \leqslant \lambda k + \epsilon$ for any $\gamma \in \Gamma$. So $\{\gamma(z_i) \cdot x_0\} \cdot B_k \subset \{\gamma(z_i) \cdot x_0\}[\lambda k + \epsilon]$. Since $\lim_{i\to\infty}\{z_i \cdot \gamma\} = z \cdot \gamma$ by continuity,

$$z \cdot B_k \subset \overline{\{\gamma(z_i) \cdot x_0\}[\lambda k + \epsilon]} \cap (Z - Z^0) \subset L.$$

Letting k increase, we see that $z \cdot \Gamma \subset L$.

This theorem indicates outlines of sets which must be very close to being boundedly saturated, and in many cases they are such. An example might be our own application which comes next or the case of a uniform lattice Γ acting on the symmetric space compactified by the ideal boundary. The theorem correctly suggests that each ideal point fixed by the trivial extension of the right action of Γ is also boundedly saturated.

Let us now return to the situation with $Z = \nu N$ where $Z^0 = N$ is given the Γ-invariant metric defined above. Theorem 5.3.1 suggests that the cells from Proposition 4.2.1 might be good candidates for boundedly saturated subsets of τN.

5.4. BOUNDED SATURATION: THE PROOF

Now we formally confirm the guess we made in Section 5.3. In the case Z^0 is a Lie group which acts on itself by left multiplication and the chosen metric is left invariant, Theorem 5.3.1 has a much stronger analogue.

THEOREM 5.4.1. *Each right Z^0-orbit in $Z \backslash Z^0$ is boundedly saturated.*

Proof. Let $z \in \partial Z^0$ and $C \subset Z$ be a closed subset such that $C \cap (Z \backslash Z^0) \subset z \cdot Z^0$. Suppose there exists a number D with the property that $\overline{(C \cap Z^0)[D]} \backslash (z \cdot Z^0) \neq \emptyset$. Then there is a sequence $\{y_i\} \subset (C \cap Z^0)[D]$ with the limit $\lim_{i\to\infty}\{y_i\} = y \notin z \cdot Z^0$. For each y_i choose $z_i' \in \{z_i\}$ such that $d(y_i, z_i') \leqslant D$. Then $\lim_{i\to\infty}\{z_i'\} = \lim_{i\to\infty}\{z_i\}$. Also there are elements $b_i \in Z^0$ such that $z_i' = y_i \cdot b_i$, they satisfy

$d(I, b_i) = d(y_i, y_i \cdot b_i) = d(y_i, z_i') \leqslant D$. This infinite sequence has a cluster b in the D-ball $B_D \subset Z^0$. From the continuity of the right action we have

$$z \cdot b = \lim_{i\to\infty}\{z_i\} \cdot b = \lim_{i\to\infty}\{z_i \cdot b_i\} = \lim_{i\to\infty}\{y_i\} = y$$

which contradicts the assumption.

Since every subset of τN is right N-invariant, we have

COROLLARY 5.4.2. *Each cell in the cellular decomposition of τN from Section 4.2 is boundedly saturated.*

5.5. CUBICAL CELLULAR DECOMPOSITIONS

Let $I^r = [-1, 1]^r$ be the r-dimensional cube embedded in $\mathbb{R}^r$. It has 2^n vertices indexed by various r-tuples with entries either 1 or -1. Let us denote this set by $V_{(-1)}$. We also say that $V_{(-1)}$ is derived from $I_{(-1)} = \{\pm 1\}$ and write this as $V_{(-1)} = I^r_{(-1)}$. Now define the following subsets of I:

$$I_{(0)} = \{-1, 0, 1\}, \qquad I_{(1)} = \{-1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1\}, \ldots,$$

where

$$I_{(i)} = \left\{-1, \ldots, \frac{k}{2^i}, \frac{k+1}{2^i}, \ldots, 1\right\}, \quad k \in \mathbb{Z}, \quad -2^i \leqslant k \leqslant 2^i,$$

for $i \in \mathbb{N}$. We also get the corresponding derived subsets of I^r:

$$V_{(0)}, V_{(1)}, \ \ldots \ , V_{(i)} = \{v_i(s_1, \ldots, s_r)\} = I^r_{(i)}, \ldots,$$

where

$$v_i(s_1, \ldots, s_r) \overset{\text{def}}{=} \left(\frac{s_1}{2^i}, \ldots, \frac{s_r}{2^i}\right), \quad s_j \in \mathbb{Z}, \quad -2^i \leqslant s_j \leqslant 2^i.$$

At each stage $V_{(i)}$ is the set of vertices of the obvious cellular decomposition of I^r, where the top dimensional cells are r-dimensional cubes with the j-th coordinate projection being an interval

$$\left[\frac{k_j}{2^i}, \frac{k_j+1}{2^i}\right] \subset I, \quad 1 \leqslant j \leqslant i.$$

These cells can be indexed by the n-tuples $\{(k_1, \ldots, k_j, \ldots, k_r) : -2^i \leqslant k_j < 2^i\}$, the coordinates of the lexicographically smallest vertex, $2^{(i+1)r}$ of the r-tuples at all.

These decompositions behave well with respect to the sequence of collapses from Section 4.2 and induce cellular decompositions of the result from the (-1)-st derived decomposition of I^r and the corresponding CW-structure in B^r. We will refer to this isomorphism of CW-structures as $\Upsilon\colon \partial B^r \to \tau N$.

There are cubical analogues of links and stars of the usual simplicial notions. Thus the *star* of a vertex is the union of all cells which contain the vertex in the boundary. The *open star* is the interior of the star. For the i-th derived cubical decomposition, the open star of the vertex $v_i(s_1, \ldots, s_r)$ will be denoted by $\mathrm{Star}^{\mathrm{o}}(v_i(s_1, \ldots, s_r))$. These sets form the *open star covering* of I^r.

By *vertices* in δN we mean the image $\Upsilon\rho(V_{(n)} \cap \partial I^r)$. Let $v \in \Upsilon\rho(V_{(n)} \cap \partial I^r)$ then

$$\mathrm{Star}^{\mathrm{o}}((\Upsilon\rho)^{-1}(v) \cap V_{(n)}) = \bigcup_{\substack{v_n \in V_{(n)} \\ \Upsilon\rho(v_n)=v}} \mathrm{Star}^{\mathrm{o}}(v_n)$$

is an open neighborhood (the open star) of $(\Upsilon\rho)^{-1}(v)$, and, in fact,

$$\mathrm{Star}^{\mathrm{o}}_n(v) \overset{\mathrm{def}}{=} \Upsilon\rho(\mathrm{Star}^{\mathrm{o}}(\rho^{-1}\Upsilon^{-1}(v) \cap V_{(n)}))$$

is an open neighborhood of v which we call the *open star* of v. The map $\Upsilon\rho$ is bijective in the interior of I^r, so $\mathrm{Star}^{\mathrm{o}}_n(v)$ can be defined by the same formula for $v \in \Upsilon\rho(V_{(n)} \cap \operatorname{int} I^r)$.

6. Borel–Serre Enlargements and their Compactification

6.1. THE BOREL–SERRE ENLARGEMENT

Let $G = G(\mathbb{Q})$ be a semi-simple algebraic subgroup of $\mathrm{GL}_n(\mathbb{Q})$ and Γ be an arithmetic subgroup of G. It is a lattice the real Lie group $G(\mathbb{R})$ and acts (not cocompactly) on the symmetric space of maximal compact subgroups $X = G(\mathbb{R})/K$ so that X is a model for $E\Gamma$ if Γ is torsion-free. Borel and Serre ([8]) form a contractible enlargement $\bar{X} \supset X$ which depends only on the $\mathbb{Q}$-structure of G so that the action of Γ extends to $\bar{X}$. The space $\bar{X}$ is another model for $E\Gamma$ but now the action is cocompact.

We discuss the two cases $k = \mathbb{Q}$ and $\mathbb{R}$ simultaneously. Denote by $\mathcal{P}_k(G)$ the set of parabolic k-subgroups of G. Let $P \in \mathcal{P}_k(G)$, and let $\widehat{S}_P$ denote the maximal k-split torus of the center $\widehat{C}_P$ of the Levi quotient $\widehat{L}_P$, and $\widehat{A}_P = \widehat{S}_P(\mathbb{R})^0$. (An object associated to the reductive Levi quotient $\widehat{L}_H$ rather than the group H itself will usually indicate this by wearing a ‘hat’.) To each $x \in X$ is associated the Cartan involution θ_x of G that acts trivially on the corresponding maximal compact subgroup (see [8]). There is a unique θ_x-stable lift $\tau_x\colon \widehat{L}_P(\mathbb{R}) \to P(\mathbb{R})$ which gives the θ_x-stable lifting $A_{P,x} = \tau_x(\widehat{A}_P)$ of the subgroup $\widehat{A}_P$.

DEFINITION 6.1.1. The *geodesic action* of $\widehat{A}_P$ on X is given by $a \circ x = a_x \cdot x$, where $a_x = \tau_x(a) \in A_{P,x}$ is the lifting of $a \in \widehat{A}_P$.

Let $\widehat{T}_G$ be a maximal k-split torus of $\widehat{L}_G/\widehat{C}_G$ and $\widehat{\Delta}_G$ be the system of positive simple roots with respect to $\widehat{T}_G$. There is a lattice isomorphism $\Theta \mapsto P_\Theta$ between the power set of $\widehat{\Delta}_G$ and the set of standard parabolic k-subgroups of G.

Now X can be viewed as the total space of a principal $\widehat{A}_P$-bundle under the geodesic action, and $\widehat{A}_P$ can be openly embedded into $(\mathbb{R}_+^*)^{\mathrm{card}(\widehat{\Delta}-\Theta(P))}$ via $\widehat{A}_P \mapsto \mathbb{R}^{\mathrm{card}(\widehat{\Delta}-\Theta(P))}$. Let $\bar{A}_P$ be the 'corner' consisting of $\widehat{A}_P$ together with positive card$(\widehat{\Delta} - \Theta(P))$-tuples where the entry ∞ is allowed with the obvious topology making it diffeomorphic to $(0,\infty]^{\mathrm{card}(\widehat{\Delta}-\Theta(P))}$. Now $\widehat{A}_P$ acts on $\bar{A}_P$, and the *corner* $X(P)$ associated to P is the total space of the associated bundle $X \times_{\widehat{A}_P} \bar{A}_P$ with fiber $\bar{A}_P$. Denote the common base of these two bundles by $e(P) = \widehat{A}_P \backslash X$. In particular, $e(G^0) = X$.

DEFINITION 6.1.2. The *Borel–Serre enlargement*

$$\bar{X}_k = \bigsqcup_{P \in \mathcal{P}_k(G)} e(P)$$

has a natural structure of a manifold with corners in which each corner $X(P) = \bigsqcup_{Q \supset P} e(Q)$ is an open submanifold with corners. The action of $Q(k)$ on X extends to the enlargement $\bar{X}_k$. The faces $e(P)$, $P \in \mathcal{P}_k(G)$, are permuted under this action.

Remark 6.1.3. When B is a Borel $\mathbb{R}$-subgroup of G, we have the Iwasawa decomposition $G(\mathbb{R}) = K \cdot A_B \cdot N_B(\mathbb{R})$ (see [46, Theorem 3.9]). Then $X \approx A_B \cdot N_B(\mathbb{R})$, and the geodesic action of A_B on X coincides with multiplication. The quotient $e(B)$ can be viewed as the underlying space of the nilpotent group $N_B(\mathbb{R})$.

The main result of Borel and Serre about this construction is that $\bar{X}_{\mathbb{Q}}$ is contractible, the action of Γ on $\bar{X}_{\mathbb{Q}}$ is proper, and the quotient $\Gamma\backslash\bar{X}_{\mathbb{Q}}$ is compact. So, indeed, $\bar{X}_{\mathbb{Q}}$ is the new $E\Gamma$ we can use. The space $\widehat{X}$ to be constructed in Section **??** will compactify $\bar{X}_{\mathbb{Q}}$.

DEFINITION 6.1.4 (Zucker [59]). Let $q_P \colon X \to e(P)$ denote the bundle map. For any open subset $V \subset e(P)$ a cross-section σ of q_P over V determines a translation of V from the boundary of $\bar{X}_k$ into the interior X. For any $t \in \widehat{A}_P$ put $\widehat{A}_P(t) = \{a \in \widehat{A}_P : \ a^\alpha > t^\alpha$ for all $\alpha \in \widehat{\Delta} - \Theta(P)\}$. For any cross-section $\sigma(V)$, a set of the form $\widehat{W}(V,\sigma,t) = \widehat{A}_P(t) \circ \sigma(V)$ is called an *open set defined by geodesic influx from V into X.*

There is a natural isomorphism $\mu_\sigma \colon \widehat{A}_P(t) \times V \cong \widehat{W}(V,\sigma,t)$ which extends to a diffeomorphism $\bar{\mu}_\sigma \colon \bar{A}_P(t) \times V \cong W(V,\sigma,t)$. Now $W(V,\sigma,t)$ is a neighborhood of V in $\bar{X}$ with $\bar{\mu}_\sigma(\{(\infty,\dots,\infty)\} \times V) = V$. This is an *open neighborhood defined by geodesic influx from V into X.*

EXAMPLE 6.1.5 ($\bar{X}(\mathrm{SL}_2)$). The hyperbolic plane X can be thought of as the open unit disk $\mathbb{E}$ in $\mathbb{C}$ or as the upper half-plane $\mathbb{H}$. Elements of $\mathrm{SL}_2(\mathbb{Q})$ act on $\mathbb{H}$ from the left as Möbius transformations. The action extends to the hyperbolic boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. The models $\mathbb{E}$ and $\mathbb{H}$ are related via the biholomorphic Cayley mapping $\mathbb{H} \to \mathbb{E}$, $z \mapsto (z-i)/(z+i)$. The *rational points* on the unit circle $\partial\mathbb{E}$ are the image of $\mathbb{Q} \subset \mathbb{R} \subset \partial\mathbb{H}$. The proper $\mathbb{Q}$-parabolic subgroups P are the stabilizers of the rational points p in $\partial\mathbb{E}$. All of them are Borel subgroups.

For each P the positive reals $\lambda \in \mathbb{R}_+$ act on X by translations of magnitude $\log\lambda$ along hyperbolic geodesics in the direction of the cusp p. This is the geodesic action. Each geodesic γ can be completed to a half-line by adding a limit point e_γ in the positive direction of the $\mathbb{R}_+$-action which extends trivially to e_γ. Now $X(P) = X \cup e(P)$, where $e(P)$ is a copy of $\mathbb{R}$ 'at' p which parametrizes the geodesics converging to p, and $\bar{X} = \bigcup_P X(P)$, where $P \in \mathcal{P}_{\mathbb{Q}}(\mathrm{SL}_2)$.

Given a point and an open interval $y \in V \subset e(P)$, the restriction of a cross-section of the principal bundle $X \to e(P)$ to V determines an open neighborhood W of y in $X(P)$ defined by geodesic influx from V into X, i.e., W consists of all points on geodesics connecting the image of the cross-section to V including the latter but not the former. This description makes it clear that $\bar{X}$ is a Hausdorff space. Every $g \in G$ acts as a Möbius transformation on X and sends a geodesic converging to a rational point to another hyperbolic geodesic. If $g \in \Gamma \subset \mathrm{SL}_2(\mathbb{Q})$ then the new geodesic converges to a rational point and thus defines $g \cdot y \in \bar{X}$.

6.2. COMPACTIFICATION OF $E\Gamma$

The construction performed here can be compared to other compactifications of a symmetric space X of Martin, Satake and Furstenberg, Karpelevič, and the ideal compactification. Our $\widehat{X}$ also contains $\bar{X}$ as an open dense subspace. This gives it more algebraic flavor than is present in (at least the original formulations of) the other constructions.

The corner $X(P)$ can be constructed for any parabolic subgroup of G defined over $\mathbb{R}$ (see [8]). This means that instead of $\bar{X} = \bar{X}_{\mathbb{Q}}$ we can obtain a larger space $\bar{X}_{\mathbb{R}} = \bigcup_P X(P)$, where P ranges over all proper $\mathbb{R}$-parabolic subgroups. In general, there may appear complications in the way $\bar{X}_{\mathbb{Q}}$ and $\bar{X}_{\mathbb{R}}$ fit together arising, for example, from the fact that the $\mathbb{Q}$-rank of G may not be equal to the $\mathbb{R}$-rank. Restricting our attention to the case of $\mathrm{rank}_{\mathbb{R}} G = 1$ (which we assume from now on) avoids such phenomena.

For an arithmetic subgroup Γ of $G(\mathbb{Q})$ and any Borel subgroup $B \in \mathcal{B}_{\mathbb{Q}}$, $\Gamma_B = \Gamma \cap B(\mathbb{Q})$ is the stabilizer of $e(B)$. If we write the Langlands decomposition as $B(\mathbb{R}) = M(\mathbb{R}) \cdot A \cdot N(\mathbb{R})$ then $\Gamma_B \subset M(\mathbb{R}) \cdot N(\mathbb{R})$. Since in our case Γ is torsion-free, $\Gamma_B = \Gamma_N = \Gamma \cap N(\mathbb{Q})$.

PROPOSITION 6.1.6. *Γ_N is a uniform nilpotent lattice in N.*

This is precisely the property called *admissibility* in [30], and our Γ are proved to be always admissible in [30, Theorem 5.3].

We start by compactifying each $e(B)$, $B \in \mathcal{B}_{\mathbb{R}}$, $\Gamma \cap B(\mathbb{R})$-equivariantly, then provide the new points with certain neighborhoods which will form a part of the basis for the topology on $\widehat{X}$. Recall Remark 6.1.3. The lattice $\Gamma \cap N$ acts on N via left multiplication. We refer to [33, Lemma (7.8)] and the preceeding discussion for the verification that this is, in fact, the action of Γ_B on the stratum $e(B)$. Thus the material of Section 4 becomes relevant, and $e(B_0)$ corresponding to the standard Borel subgroup B_0 may, indeed, be compactified Γ_{B_0}-equivariantly by τN.

The conjugation action of $G(\mathbb{R})$ permutes the Borel–Serre strata $e(B)$, so $G(\mathbb{R})$ also acts on the disjoint union of the compactifications $\nu(B) = \nu(e(B))$, i.e., on

$$\delta X \overset{\text{def}}{=} \nu(B_0) \times_{B_0(\mathbb{R})} G(\mathbb{R}).$$

Warning. δX comes with the identification topology which we are going to use in the ensuing construction, but it will not be the subspace topology induced from the resulting topology on $\widehat{X}$.

DEFINITION 6.1.7. $\widehat{X} = \bar{X}_{\mathbb{R}} \cup \delta X = X \sqcup \delta X$.

The topology is introduced à la Bourbaki. We are referring to

PROPOSITION 6.1.8 ([9, Proposition 1.2.2]). *Let X be a set. If to each $x \in X$ there corresponds a set $\mathcal{N}(x)$ of subsets of X such that* (1) *every subset of X containing one from $\mathcal{N}(x)$ itself belongs to $\mathcal{N}(x)$,* (2) *a finite intersection of sets from $\mathcal{N}(x)$ belongs to $\mathcal{N}(x)$,* (3) *the element x belongs to every set in $\mathcal{N}(x)$,* (4) *for any $N \in \mathcal{N}(x)$ there is $W \in \mathcal{N}(x)$ such that $N \in \mathcal{N}(y)$ for every $y \in W$, then there is a unique topology on X such that, for each $x \in X$, $\mathcal{N}(x)$ is the set of neighborhoods of x.*

By a *neighborhood* of a subset A in a topological space they understand any subset which contains an open set containing A.

The space $\bar{X}_{\mathbb{R}}$ is the $\mathbb{R}$-Borel–Serre construct and has the topology in which each corner $X(B)$ is open. For $y \in \bar{X}_{\mathbb{R}}$ let $\mathcal{N}(y) = \{\mathcal{O} \subset \widehat{X} : \mathcal{O}$ contains an open neighborhood of y in $\bar{X}_{\mathbb{R}}\}$.

Notation. Given an open subset $U \subset \nu(B)$, let $\mathcal{O}(U) = q_B^{-1}(V)$, the total space of the restriction to $V = U \cap e(B)$ of the trivial bundle q_B over $e(B)$ with fiber A_B. If U is any open subset of δX, let

$$\mathcal{O}(U) = \bigcup_{B \in \mathcal{P}_{\mathbb{R}}} \mathcal{O}(U \cap e(B)).$$

In either case define $\mathcal{C}(U) = \{z \in \bar{X}_{\mathbb{R}}$: there is $\mathcal{O} \in \mathcal{N}(z)$ such that $\mathcal{O} \cap X \subset \mathcal{O}(U)\} \cup \{z \in \delta X \backslash \bar{X}_{\mathbb{R}}$: there is an open $U' \subset \delta X$ such that $z \in U'$ and $\mathcal{O}(U') \subset \mathcal{O}(U)\}$.

Now for $y \in \delta X \backslash \bar{X}_{\mathbb{R}}$, let $\mathcal{N}(y) = \{\mathcal{O} \subset \widehat{X}$: there is an open set $U \subset \delta X$ containing y with $\mathcal{C}(U) \subset \mathcal{O}\}$. This defines a system of neighborhoods $\mathcal{N}(y)$ for any $y \in \widehat{X}$. For a subset $\mathcal{S} \subset \widehat{X}$ let $\mathcal{N}(\mathcal{S}) = \{\mathcal{O} \subset \widehat{X} : \mathcal{O} \in \mathcal{N}(y)$ for every $y \in \mathcal{S}\}$ and call $\mathcal{S}$ *open* if $\mathcal{S} \in \mathcal{N}(\mathcal{S})$. The following lemma is elementary.

LEMMA 6.1.9. (1) *If $B \in \mathcal{B}_{\mathbb{R}}$ and $U \subset \nu(B)$ is an open subset then $\mathcal{C}(U)$ is open in $\widehat{X}$.* (2) *If $B \in \mathcal{B}_{\mathbb{R}}$, and $U_1, U_2 \subset \nu(B)$ are open subsets, then $\mathcal{C}(U_1) \cap \mathcal{C}(U_2) = \mathcal{C}(U_1 \cap U_2)$.*

THEOREM 6.1.10. *The open subsets of $\widehat{X}$ form a well-defined topology in $\widehat{X}$.*

Proof. We need to check that the four characteristic properties from Proposition 6.1.8 are satisfied by $\mathcal{N}(x)$, $x \in \widehat{X}$. (1) and (3) are clear from definitions. (2) follows from Lemma 6.1.9 (2). Given any $N \in \mathcal{N}(x)$, $x \in \tau(B)$, there is $U \in \nu(B)$ such that $\mathcal{C}(U) \subset N$. Take $W = \mathcal{C}(U)$. By Lemma 6.1.9 (1), $N \in \mathcal{N}(y)$ for any $y \in W$. Thus (4) is also satisfied.

Remark 6.1.11. If Γ is an arithmetic subgroup of $G(\mathbb{Q})$, it is immediate that this compactification is Γ-equivariant. In fact, the action of $G(\mathbb{R})$ on X extends to $\widehat{X}$ which is in contrast to the fact that this action does not extend to $\bar{X}$.

EXAMPLE 6.1.12 (Arithmetic Fuchsian Groups). Consider an arbitrary proper parabolic $\mathbb{R}$-subgroup P of $G = \mathrm{SL}_2$. It acts on X just as the $\mathbb{Q}$-subgroups stabilizing a point $p(P)$ in $\partial\mathbb{E}$, i.e., P permutes geodesics abutting to $p(P)$. Attach a line at $p(P)$ parametrizing these geodesics; this is the general construction of corners $X(P)$ from ([8, Section 5]) in the case $k = \mathbb{R}$. If P fixes a rational point then $X(P) = \bar{X}_{\mathbb{Q}}$. Complete each stratum: now $\nu(P) = e(P) \cup \{-\infty, +\infty\}$. The resulting set is $\widehat{X}$ in which every $X(P)$ is declared to be open. So typical open neighborhoods of $z \in e(P)$ in $\widehat{X}$ are the open neighborhoods of z in $X(P)$. Given a line $e(P)$ and one of its endpoints y, a typical open neighborhood of y consists of

- y itself and an open ray in $e(P)$ asymptotic to y,
- an open (Euclidean) set U in $\mathbb{E}$ bounded by the hyperbolic geodesic γ abutting to $p(P)$ representing the origin of the ray in $e(P)$—the one which is the union of geodesics representing other points of the ray,
- points in various $e(B)$, $B \in \mathcal{P}_{\mathbb{R}}$, such that $p(B)$ is on the arc in $\partial\mathbb{E}$ connecting $p(P)$ with $p(R)$, the opposite end of γ, which are represented by geodesics with a subray inside U,
- each endpoint of the corresponding $\nu(B)$ if $B \neq P$, R, and
- the endpoint of $\nu(R)$ which is the limit of a ray in $e(R)$ contained in the set from (3).

With the topology on $\widehat{X}$ generated as above, the subspace $X \subset \widehat{X}$ has the hyperbolic metric topology, and $\delta X = \widehat{X} - X$ is simply $S^1 \times I$ with an analogue of the lexicographic order topology ([52, Exercise 48]). In terms of the description of the lexicographic ordering on the unit square $I \times I$ given in [52], the analogue we refer to is the quotient topology on $S^1 \times I$ associated to the obvious identification $(0,y) \sim (1,y)$ for all $y \in I$. In particular, δX is compact but not separable and, therefore, not metrizable.

7. Topological Properties of $\widehat{X}$

7.1. SOME GEOMETRIC PROPERTIES OF $\widehat{X}$

The space $\widehat{X}$ is not metrizable and, therefore, has no geometry in the usual sense. On the other hand, the geometry of the spherical compactification εX with the ideal boundary ∂X is well understood (see [4]). The goal of this section is to relate $\mathrm{Cov}^{\mathrm{s}}\,\widehat{X}$ to $\mathrm{Cov}^{\mathrm{s}}(\varepsilon X)$.

The first two lemmas are proved in the following generality: let G be a semi-simple linear algebraic $\mathbb{Q}$-group with equal $\mathbb{Q}$- and $\mathbb{R}$-ranks.

LEMMA 7.1.1. *Let $q_B\colon X \to e(B)$ be the Borel–Serre bundle associated to a minimal parabolic subgroup $B \in \mathcal{B}_{\mathbb{R}}(G)$. Consider a compact set $C \subset e(B)$ and the restriction of q_B to C with the total space $q_B|C$. Then the closure of $q_B|C$ in εX consists of the union of $q_B|C$ with apartments at infinity ∂A, where $A = q_B^{-1}(c)$ for some $c \in C$.*

Proof. Let $y \in \mathrm{cl}(q_B|C)$ and $\{y_i\}$ be a sequence of points in $q_B|C$ converging to y. If $y \in X$ then $y \in q_B|C$ by compactness of C.

Consider $y \in \mathrm{cl}(q_B|C)\backslash X$, pick a section σ of q_B, and introduce the following notation: $\bar{y}_i = \sigma(q_B(y_i))$, $z = \lim_{i\to\infty} q_B(y_i)$, $\bar{z} = \sigma(z)$. Let γ be the unique unit speed geodesic ray from $\bar{z}$ asymptotic to y, γ_i be the unit speed geodesic from $\bar{z}$ to y_i, and ρ_i be the geodesic from $\bar{y}_i$ to y_i.

If λ_i is a sequence of geodesic rays in X, it is said to *converge* to a geodesic ray λ if $\lambda(t) = \lim_{i\to\infty} \lambda_i(t)$ for every $t \in [0,\infty)$. The geodesic segments γ_i (or their extensions) do converge to γ (see [4], 3.2), i.e., $\lim_{i\to\infty} d(\gamma_i(t),\gamma(t)) = 0$. On the other hand, we can also claim that

$$\lim_{i\to\infty} d(\rho_i(t),\gamma_i(t)) = 0.$$

Proof of the claim. Let ρ_i^- and γ_i^- be the unit speed geodesics from y_i to $\bar{y}_i$ and $\bar{z}$ respectively, then $\rho_i(t) = \rho_i^-(d(y_i,\rho_i(t)))$, $\gamma_i(t) = \gamma_i^-(d(y_i,\gamma_i(t))) = \gamma_i^-(d(y_i,\rho_i(t)) + \delta_i)$, $\delta_i \in \mathbb{R}$. Let $M = \min\{d(y_i,\bar{y}_i), d(y_i,\bar{z})\}$. Without a loss of generality we can assume that $M = d(y_i,\bar{y}_i)$ so that $\delta_i \geq 0$. The geodesic bicombing of X is bounded, so there are constants k_1 and k_2 such that $d(\rho_i^-(s),\gamma_i^-(s)) \leqslant k_1 d(\bar{y}_i,\bar{z}) + k_2$ for any $0 \leqslant s \leqslant M$. In fact, we can do better and use $k_1 = 1$, $k_2 = 0$. Indeed, recall Toponogov's definition of nonpositive curvature ([4]). Now

$$\begin{aligned} d(\rho_i(t), \gamma_i(t+\delta_i)) &\leqslant d(\bar{y}_i, \gamma_i^-(M)) \\ &\leqslant d(\bar{y}_i', \gamma_i^-(M)') \leqslant d(\bar{y}_i', \bar{z}') = d(\bar{y}_i, \bar{z}), \end{aligned}$$

where the primes denote the corresponding points in the comparison triangles. By the triangle inequality,

$$d(\gamma_i(t+\delta_i), \gamma_i(t)) = |\delta_i| = |d(y_i, \bar{y}_i) - d(y_i, \bar{z})| \leqslant d(\bar{y}_i, \bar{z}),$$

thus $d(\rho_i(t), \gamma_i(t)) \leqslant 2d(\bar{y}_i, \bar{z})$, and

$$\lim_{i\to\infty} d(\rho_i(t), \gamma_i(t)) \leqslant 2 \lim_{i\to\infty} d(\bar{y}_i, \bar{z}) = 0.$$

This proves the claim.

We finally get $\lim_{i\to\infty} d(\rho_i(t), \gamma(t)) = 0$. Since the fibers $q_B^{-1}(c)$, $c \in C$, are totally geodesic, $q_B(\rho_i(t)) = q_B(y_i)$ for all times t. So $q_B(\gamma(t)) = \lim_{i\to\infty} q_B(y_i) = z$. We conclude that $\gamma(t) \in q_B^{-1}(z)$ for all t, thus $y \in \partial q_B^{-1}(z)$, $z \in C$.

The reverse inclusion of the sets is obvious.

Recall that $e(B) \cong N_B$ (Remark 6.1.3). Derived cubical cellular decompositions of I^r with vertices $V_{(n)}$ induce cellular decompositions and open coverings of νN. The open coverings are composed of the images of open stars of vertex inverses in I^r. The map we have in mind is $\Upsilon\rho\colon I^r \to N$, where Υ is the extension of the map from Section 5.5. We fix the choice of $\Upsilon\rho$ made for B from now on. The vertices in νN are defined to be the set $\Upsilon\rho(V_{(n)})$. The stars of τN in this family of cellular decompositions form a nested sequence of regular neighborhoods of the boundary denoted by

$$\mathrm{Reg}_n(B) = \mathrm{Reg}_n(\tau N_B) \stackrel{\text{def}}{=} \bigcup_{\substack{v\in\Upsilon\rho(V_{(n)})\\ v\in\tau N_B}} \mathrm{Star}_n^{\mathrm{o}}(v).$$

Recall also the notion of geodesic influx neighborhood $W(V, \sigma, t)$ from Definition 6.1.4. Given a point $x_0 \in X$, there is a horocycle $N_B \cdot x_0$ passing through x_0 parametrizing the orbits of $\widehat{A}_B$. This defines a section $\sigma\colon e(B) \to X$ of q_B with $(\sigma \circ q_B)(x_0) = x_0$. Let us denote the corresponding geodesic influx neighborhood $W(e(B), \sigma, 0)$ by $W_B(x_0)$.

LEMMA 7.1.2. *Given any minimal parabolic subgroup $B \in \mathcal{B}_{\mathbb{R}}(G)$ and an open neighborhood U of $\nu(B)$ in $\widehat{X}$, then U contains the restriction to X of an open neighborhood $V(B)$ in εX of the corresponding Weyl chamber at infinity $W(B) \subset \partial X$.*

Proof. Since $\varepsilon(B)$ is compact, there is a neighborhood $W_B(x_0)$ of $e(B)$ contained in U. For the same reason there is an integer n large enough so that $\mathcal{C}(\mathrm{Star}^{\mathrm{o}}_n(v)) \subset U$ for every vertex $v \in \Upsilon\rho(V_{(n)} \cap \partial I^r)$. Thus

$$\tau(B) \subset \bigcup_v \mathcal{C}(\mathrm{Star}^{\mathrm{o}}_n(v)) = \mathcal{C}(\mathrm{Reg}_n(B)) \subset U.$$

It is obvious directly from the definition that $\mathcal{C}(\mathrm{Star}^{\mathrm{o}}_n(v))$ is an open neighborhood of $\mathrm{Star}^{\mathrm{o}}_n(v) \cap \tau(B)$. We obtain a new open neighborhood of $\nu(B)$ in $\widehat{X}$ by taking the union $W_B(x_0) \cup \mathcal{C}(\mathrm{Reg}_n(B)) \subset U$ which we denote by $\mathcal{V}_n(B, x_0)$ or simply $\mathcal{V}_n(B)$ when the choice of $W_B(x_0)$ is not important.

Notice that $\mathcal{O}(\mathrm{Reg}_n(B)) = \mathcal{C}(\mathrm{Reg}_n(B)) \cap X$ is the union of all chambers and walls in $q_B^{-1}(z)$, $z \in \mathrm{Reg}_n(B) \cap e(B)$, based at $\sigma(z)$. Similarly, $W_B(x_0)$ is the union of all chambers based at $\sigma(z)$, $z \in e(B)$, and asymptotic to $W(B) \subset \partial X$. So $X \backslash \mathcal{V}_n(B)$ consists of chambers and walls based at $\sigma(z)$ in the flats $q_B^{-1}(z)$, $z \in \nu(B)\backslash \mathrm{Reg}_n(B)$, and *not* asymptotic to $W(B) \subset \partial X$. This is $q_B^{-1}(\nu(B)\backslash \mathrm{Reg}_n(B))\backslash W_B(x_0)$. By Lemma 7.1.1 the closure of this set in εX consists of Weyl chambers and walls in the flats $q_B^{-1}(z)$, $z \in \nu(B)\backslash \mathrm{Reg}_n(B)$, and the corresponding apartments in ∂X excluding $W(B)$ and the chambers asymptotic to it. So, the open complement $V(B) = V_n(B)$ of this set contains $W(B)$. Finally, $V_n(B) \cap X = \mathcal{V}_n(B) \cap X \subset U$.

LEMMA 7.1.3. *Assume that* $\mathrm{rank}_{\mathbb{R}} G = 1$. *Consider a subset of* $X \cup \partial X$ *of the form* $V_n(B)$. *Then* $V_n(B) \cap \partial X$ *consists of chambers at infinity* $W(P)$ *such that* $\nu(P)$ *has a neighborhood* $N \subset \widehat{X}$ *whose restriction* $N \cap X \subset \mathcal{O}(\mathrm{Reg}_n(B))$.

Proof. First, let $P \in \mathcal{B}_{\mathbb{R}}$ have the property that $W(P) \in V_n(B) \cap \partial X$. In this case the claim reduces to finding some $\mathrm{Reg}_m(P)$ so that

$$\mathcal{O}(\mathrm{Reg}_m(P)) = \mathcal{C}(\mathrm{Reg}_m(P)) \cap X \subset \mathcal{O}(\mathrm{Reg}_n(B)).$$

This follows from the fact that the map of power sets $\phi_{B,P} \colon \mathcal{P}(e(B)) \to \mathcal{P}(e(P))$ defined by $\phi_{B,P}(\mathcal{S}) = q_P(q_P^{-1}(\mathcal{S}))$ is relatively proper (the image of a compact set is relatively compact). Let $K \subset e(B)$ be a compact subset and $\{y_i\} \subset e(P)$ a sequence such that $L = \lim_{i\to\infty}\{y_i\} \in \tau(P)$ and $q_P^{-1}(y_i)\backslash \mathcal{O}(\complement K) \neq \emptyset$ for all y_i. The endpoints of each geodesic $q_P^{-1}(y_i)$ are $W(P)$ and another point $z_i \in \complement V_n(B) \subset \partial X$. Since $V_n(B)$ is open, $z = \lim_{i\to\infty}\{z_i\} \in \complement V_n(B)$. This represents L as the class of the geodesic asymptotic to z and $W(P)$ which contradicts the hypothesis. The union of $\mathcal{O}(\mathrm{Reg}_m(P))$ and a suitable geodesic influx neighborhood of $e(P)$ is a required neighborhood N of $\nu(P)$.

Now suppose that $\nu(P)$ has a neighborhood N described in the statement. Then there is a section σ of q_P such that all Weyl chambers based at $\sigma(\xi)$, $\xi \in e(P)$, and asymptotic to $W(P)$ are contained in the neighborhood, and, therefore, miss $X\backslash V_n(B)$ completely. It is now clear that no Weyl chamber in $X\backslash V_n(B)$ is asymptotic to $W(P)$, so $W(P) \subset \mathcal{O}(\mathrm{Reg}_n(B))$.

COROLLARY 7.1.4. *Using the notation above, if the parabolic subgroup $P \in \mathcal{P}_{\mathbb{R}}$ has the property $W(P) \subset V_n(B)$ then $\nu(P) \subset \mathcal{V}_n(B)$.*

Proof. This is a corollary of the proof above. One can see immediately that $e(P) \subset \mathcal{V}_n(B)$. For $y \in \tau(P)$, $y \in \mathcal{C}(\mathrm{Reg}_m(P))$ which is of the form constructed in Lemma 7.1.3. In other words, for $P' \in \mathcal{P}_{\mathbb{R}}$ with $W(P') \subset V_m(P)$ we get $\mathcal{O}(\mathrm{Reg}_k(P')) \subset \mathcal{O}(\mathrm{Reg}_m(P))$. So $\mathcal{C}(\mathrm{Reg}_m(P)) \subset \mathcal{C}(\mathrm{Reg}_n(B))$, hence $\tau(P) \subset \mathcal{V}_n(B)$, and, finally, $\nu(P) \subset \mathcal{V}_n(B)$.

LEMMA 7.1.5. *If $\{U_P\}$, $P \in \mathcal{P}_{\mathbb{R}}$, is a collection of open sets in $\widehat{X}$ with each U_P containing $\varepsilon(P)$ whose restrictions cover $\widehat{X} - X$, then the sets $V(U_P)$ can be chosen so that they cover ∂X.*

Proof. Clear, since every point of ∂X belongs to a well defined Weyl chamber.

7.2. THE HAUSDORFF PROPERTY

The subspace $\bar{X}_{\mathbb{R}} \subset \widehat{X}$ is open, so it suffices to check the Hausdorff property for x, $y \in \delta X \backslash \bar{X}_{\mathbb{R}}$. If $x, y \in \nu(B)$ for some $B \in \mathcal{B}_{\mathbb{R}}$ then they can be separated by open neighborhoods U_x, $U_y \subset \nu(B)$ with $\mathcal{O}(U_x) \cap \mathcal{O}(U_y) = \emptyset$ which get completed to open neighborhoods $\mathcal{C}(U_x) \cap \mathcal{C}(U_y) = \emptyset$. So suppose $x \in \nu(B_1)$, $y \in \nu(B_2)$, $B_1 \neq B_2$. The points $W(B_1)$ and $W(B_2)$ are limit points of a unique apartment which projects to $x' \in e(B_1)$, $y' \in e(B_2)$. Choose $n \in \mathbb{N}$ large enough so that $x' \notin \mathrm{Reg}_n(B_1)$, then $\mathcal{C}(\mathrm{Reg}_n(B_1)) \cap \nu(B_2) = \emptyset$. Now choose $m \in \mathbb{N}$ large enough so that $\mathrm{Reg}_m(B_2) \cap q_{B_2}(\mathcal{O}(\mathrm{Reg}_n(B_1)) = \emptyset$. The existence of such m follows from the same argument as in the proof of Lemma 7.1.3. Now $\mathcal{C}(\mathrm{Reg}_n(B_1))$ and $\mathcal{C}(\mathrm{Reg}_m(B_2))$ are disjoint open neighborhoods of x and y respectively.

Remark 7.2.1. Let $\bar{X}_{\mathrm{b}}(k) = X \cup \bigcup_{B \in \mathcal{B}_k} e(B)$ where all of $X(B)$ are open. The construction from Section **??** can be performed with the Borel subgroups in any split rank linear algebraic group. The strata $e(B)$ get compactified to $\nu(B) \cong \nu N_B \subset \delta_{\mathrm{b}} X$. Denote $\bar{X}_{\mathrm{b}}(\mathbb{R}) \cup \delta_{\mathrm{b}} X$ by $\widehat{X}_{\mathrm{b}}$.

It is not true that $\widehat{X}_{\mathrm{b}}$ is always Hausdorff. This has to do with rank, and the simplest example is $\widehat{X}_{\mathrm{b}}(\mathrm{SL}_3)$. Here each maximal 2-dimensional flat consists of six chambers and six walls. Pick two walls which are in opposition: they lie on a geodesic γ through the base point and determine two walls $W(P_1)$, $W(P_2)$ at infinity. If $z_1 = q_{P_1}(\gamma) \in e(P_1)$ then let $z_1^u \in R_u P_1(\mathbb{R})$ be the first coordinate projection of $e(P_1) = R_u P_1(\mathbb{R}) \times \widehat{e}(P_1)$, where R_u denotes the unipotent radical, and $\widehat{e}(P_1)$ is the reductive Borel–Serre stratum (see [58, 59] or [33, Section 7]). The point $z_2^u \in R_u P_2(\mathbb{R})$ is defined similarly. The two points are the limits of γ in $\bar{X}$. It turns out that the points of $\{z_1^u\} \times \widehat{e}(P_1)$ and $\{z_2^u\} \times \widehat{e}(P_2)$ match bijectively in this manner.

By [8, Sections 2.8, 3.10, 5.2, 7.2(iii)], for any $P \in \mathcal{P}_{\mathbb{R}}$ the principal $R_u P(\mathbb{R})$-fibration μ_P extends to a principal fibration $\bar{\mu}_{P,\mathbb{R}}\colon \overline{e(P)}_{\mathbb{R}} \longrightarrow \overline{\widehat{e}(P)}_{\mathbb{R}}$. Since $\overline{\widehat{e}(P)}_{\mathbb{R}} =$

$\bar{X}_{\mathbb{R}}$ for $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, each level gets compactified as in Example 6.1.12. In particular, $\{z_i^u\} \times \overline{\widehat{e}(P_i)}$, $i = 1, 2$, embed in the closures of the corresponding strata. It is now easy to see that the bijective correspondence described above extends to these enlargements and to find points $y_i \in \{z_i^u\} \times (\overline{\widehat{e}(P_i)} - \widehat{e}(P_i))$ so that any two neighborhoods of y_1 and y_2 in the respective enlargements contain some points $x_i \in \{z_i^u\} \times \widehat{e}(P_i)$ which are matched. Equivalently, y_1 and y_2 are inseparable in $\widehat{X}_{\mathrm{b}}(\mathrm{SL}_3)$.

In order to construct the correct $\widehat{X}(\mathrm{SL}_3)$ and introduce a compact Hausdorff topology, one might want to compare $\widehat{X}$ with the Satake compactification of X. See Section **??** for an illustration. Complete details are contained in [32].

7.3. COMPACTNESS

Given any open subset U of $\widehat{X}$ containing $\widehat{X} - X$, since such a subset would contain $\nu(P)$ for every $P \in \mathcal{P}_{\mathbb{R}}$, its restriction to X would also contain an open neighborhood in $X \cup \partial X$ of the corresponding Weyl chamber at infinity $W(P)$ according to Lemma 7.1.2. As before, this says that $U \cap X$ is the restriction of an open subset of $X \cup \partial X$ containing ∂X. By compactness of ∂X, $U \cap X$ contains a collar on ∂X.

Now given any open covering $\mathcal{U}$ of $\widehat{X}$, let $\{U_{1,P}, \ldots, U_{k_P,P}\} \subset \mathcal{U}$ be any finite subcollection which covers the compact subspace $\nu(P)$ for $P \in \mathcal{P}_{\mathbb{R}}$. The sets $\nu(P)$ cover $\widehat{X} - X$. Since the unions $U_P = \bigcup_i U_{i,P}$ contain the corresponding $\nu(P)$ individually, they together cover $\widehat{X} - X$. We now apply Lemma 7.1.2 to find open neighborhoods $Y(P)$ of $\varepsilon(P)$ inside U_P and open neighborhoods $V(P)$ of $W(P)$ in εX which have $Y(P) \cap X = V(P) \cap X$. By Lemma 7.1.5 the sets $V(P)$, $P \in \mathcal{P}_{\mathbb{R}}$, cover ∂X.

Choose a finite subcollection $\{P_i\}$, $P_i \in \mathcal{P}_{\mathbb{R}}$, $i = 1, \ldots, m$, such that $\{V(P_i)\}$ still cover ∂X. The first paragraph shows that their union must contain a collar on ∂X. The complement of this collar in $X \cup \partial X$ is closed and contained in X, hence is compact. Let $U_{m+1}, \ldots, U_n$ be a finite collection of sets from $\mathcal{U}$ such that $U_{m+1} \cap X, \ldots, U_n \cap X$ cover the complement of the collar. Each Weyl chamber $W(P)$ is contained in at least one set $V(P_i)$. By Lemma 7.1.3 the corresponding set U_{P_i} in $\widehat{X}$ contains $\varepsilon(P)$. This means that $\{U_{P_i}\}$, $i = 1, \ldots, m$, cover $\widehat{X} - X$. Since $\bigcup_i (U_{P_i} \cap X) \supset \bigcup_i (V(P_i) \cap X)$, the sets $U_{P_1}, \ldots, U_{P_m}, U_{m+1}, \ldots, U_n$ cover $\widehat{X}$. In other words,

$$\{U_{1,P_1}, \ldots, U_{k_{P_1},P_1}, \ldots, U_{1,P_m}, \ldots, U_{k_{P_m},P_m}, U_{m+1}, \ldots, U_n\}$$

is a finite subcovering of $\mathcal{U}$.

COROLLARY 7.3.1. *The space $\widehat{X}$ is a compactification of $\bar{X}$, i.e., a compact Hausdorff space containing $\bar{X}$ as an open dense subset. In fact, the combination of the Hausdorff property and compactness makes $\widehat{X}$ normal.*

7.4. ČECH-ACYCLICITY

Since the continuous map $f\colon \widehat{X} \to \varepsilon X$ defined by

$$x \longmapsto \begin{cases} x & \text{if } x \in X, \\ W(P) & \text{if } x \in \varepsilon(P), \end{cases}$$

has contractible image and point inverses, it would be desirable to have an analogue of the Vietoris–Begle theorem for the modified Čech theory. We prove a weaker but sufficient

THEOREM 7.4.1. *If $f\colon X \to Y$ is a surjective continuous map with $f^{-1}(y)$ contractible for each $y \in Y$, and Y is Chogoshvili-acyclic for any Abelian coefficient group, then*

$$\check{f}\colon \check{h}(X; KR) \longrightarrow \check{h}(Y; KR)$$

is a weak homotopy equivalence. So both X and Y are Čech-acyclic.

The proof is an amalgam of results from [5, 6, 20, 36, 51]. The construction of Chogoshvili is the one we have sketched in Section **??**; it extends the Steenrod homology theory on the subcategory of compacta. Berikashvili ([5, Theorem 2]) proved the uniqueness of such an extension

$$h_*(\_\,,\_)\colon \text{CompHaus}^2 \longrightarrow \text{AbGroups}$$

when it satisfies the following three axioms.

Axiom A. If (X, K) is a compact Hausdorff pair then the projection $(X, K) \to (X/K, \text{point})$ induces an isomorphism $h(X, K) \to h(X/K, \text{point})$.

Axiom B. For the diagram $\{(S^n_\alpha, \text{point}), \pi_{\alpha\beta}\}$, where S^n_α is a finite bouquet of n-dimensional spheres and $\pi_{\alpha\beta}\colon S^n_\alpha \to S^n_\beta$ is a mapping sending each sphere of the bouquet either to the distinguished point or homeomorphically onto a sphere in the target, there are isomorphisms

$$h_i\left(\varprojlim_\alpha \{(S^n_\alpha, \text{point}), \pi_{\alpha\beta}\}\right) \cong \varprojlim_\alpha \{h_i(S^n_\alpha, \text{point}), \pi_{\alpha\beta}\}.$$

Let $\mathcal{E} = (E_1, \ldots, E_k) \in \mathcal{A}$ be a finite decomposition as in Section **??**. Let $N_\mathcal{E}$ denote the nerve of the finite closed covering $\overline{\mathcal{E}} = (\overline{E_1}, \ldots, \overline{E_k})$. Then

$$N(X) \overset{\text{def}}{=} \varprojlim_{\mathcal{E}\in\mathcal{A}} N_\mathcal{E}, \qquad N_p(X) \overset{\text{def}}{=} \varprojlim_{\mathcal{E}\in\mathcal{A}} N^p_\mathcal{E},$$

where K^p denotes the p-th skeleton of the simplicial complex K. There is a unique continuous map

$$\omega\colon N(X) \longrightarrow X$$

determined by the condition that if $y = \{y_{\mathcal{E}}\} \in N(X)$, $y_{\mathcal{E}} \in N_{\mathcal{E}}$, and $\sigma_{y,\mathcal{E}} = (E_1, \ldots, E_i)$ is the minimal simplex in $N_{\mathcal{E}}$ containing $y_{\mathcal{E}}$, the *carrier* of $y_{\mathcal{E}}$, then $\omega(y) \in \overline{E_1} \cap \cdots \cap \overline{E_i}$. Indeed, $\bigcap_{\sigma_{y,\mathcal{E}}} \overline{E_1} \cap \cdots \cap \overline{E_i} \neq \emptyset$, and uniqueness follows from the Hausdorff property of X.

Axiom C. The natural homomorphism

$$\underset{p\geqslant 0}{\underrightarrow{\operatorname{colim}}}\, h_*(N_p(X)) \longrightarrow h_*(X)$$

induced by ω is an isomorphism.

In [36] Inassaridze derives the Vietoris–Begle theorem for such a theory with coefficients in the category of Abelian groups. His theorem requires point inverses to be homologically trivial. Applying the theorem to the map $f\colon X \to Y$, we get an isomorphism

$$H_*(f;A)\colon H_*(X;A) \cong H_*(Y;A)$$

of Chogoshvili homology groups for any $A \in$ ABGROUPS. So X itself is Chogoshvili-acyclic for any Abelian group of coefficients.

Now the main tool of Berikashvili in [5, 6] is the following characterization ([6, Theorems 3.1, 3.4]).

THEOREM 7.4.2. *A generalized homology theory k_* on the category of compact Hausdorff spaces satisfies Axioms A, B, and C, if and only if there exists a functorial convergent Atiyah–Hirzebruch spectral sequence with*

$$E^2_{p,q} = H_p(X; k_q(\text{point})) \Longrightarrow k_{p+q}(X).$$

When X is Chogoshvili-acyclic for all Abelian coefficient groups, this sequence collapses at the E_2-term with just the right entries in the 0-th column to make X k-acyclic. Axiom A is satisfied by any Steenrod theory. To complete the proof of Čech-acyclicity of $\widehat{X}$, it suffices to verify that Axioms B and C hold for the modified Čech theory with coefficient spectrum $K(R)$ (cf. [51]).

LEMMA 7.4.3. *Let $\{X_\alpha\}$ be an inverse system of compact Hausdorff spaces with $X = \lim_\alpha X_\alpha$. Then there is a spectral sequence with*

$$E^2_{p,q} = \underset{\alpha}{\underleftarrow{\lim}}^{\mathrm{p}}\, \check{h}_q(X_\alpha; KR)$$

converging to $\check{h}_*(X;KR)$.

Proof. This is identical to the proof of Theorem 8.5.1 from [22]. Observe that

$$N(\mathrm{Cov}^{\mathrm{s}}\, X) \cong \{N(\mathrm{Cov}^{\mathrm{s}}\, X_\alpha)\} \in \text{PRO-S-SETS}.$$

The lemma follows from the Bousfield–Kan spectral sequence (Theorem 1.1.5) applied to $\{N(\mathrm{Cov}^{\mathrm{s}}\, X_\alpha)\}$ viewed as an object in the category PRO-(PRO-S-SETS).

Apply the lemma to the system of wedges of spheres and notice that

$$\varprojlim_{\alpha}{}^{\mathrm{p}}\, \check{h}_q(S^n_\alpha; KR) = 0 \quad \text{for } p > 0$$

by Corollary 1.2 of [6] or [37]. Since also

$$\varprojlim_{\alpha}{}^{\mathrm{o}}\, \check{h}_q(S^n_\alpha; KR) = \varprojlim_{\alpha} \check{h}_q(S^n_\alpha; KR),$$

the isomorphism between the E^2- and E^∞-terms is the one required in Axiom B.

LEMMA 7.4.4. *For each integer $\varpi \leqslant 0$ there is a spectrum $K^\varpi(R)$ with*

$$K^\varpi_i(R) = \begin{cases} \pi_i(K^\varpi R) = 0 & \textit{for } \ i < \varpi, \\ K^\varpi_i(R) = K_i(R) & \textit{for } \ i > \varpi. \end{cases}$$

Proof. $K^\varpi R = \Omega^{-\varpi}\mathrm{Spt}(i\mathcal{C}_{-\varpi}(R))$. See Section 1.2 for notation.

Apply Lemma 7.4.3 to each $N_s(X)$, $s \geqslant 0$, and homology theory $\check{h}_*(\_; K^\varpi R)$, getting spectral sequences with

$$E^2_{p,q} = \varprojlim_{\mathcal{E}\in\mathcal{A}}{}^{\mathrm{p}}\, \check{h}_q(N^s_{\mathcal{E}}; K^\varpi R).$$

Now, for any finite complex C

$$\check{h}_n(C^p; K^\varpi R) = \check{h}_n(C; K^\varpi R)$$

for $p > n - \varpi$. So each entry in the E^2-term associated with $N_s(X)$ with q-coordinate $< s + \varpi$ coincides with the corresponding entry in the E^2-term associated with $N(X)$. Passing to the limit as $s \to \infty$ we see that the natural map

$$t(X)\colon \varinjlim_{p\geqslant 0} \check{h}_*(N_p(X); K^\varpi R) \longrightarrow \check{h}_*(N(X); K^\varpi R)$$

is an isomorphism.

Notice that the natural homomorphism from Axiom C factors as

$$\varinjlim_{p\geqslant 0} \check{h}_*(N_p(X); K^\varpi R) \xrightarrow{t(X)} \check{h}_*(N(X); K^\varpi R) \xrightarrow{\omega_*} \check{h}_*(X; K^\varpi R).$$

It remains to show that ω_* is an isomorphism.

Recall a construction due to Eldon Dyer. Let $\mathcal{U} \in Cov X$ and set

$$L_{\mathcal{U}}(X) = \{(y, x) \in N\mathcal{U} \times X : x \in \mathrm{cl}(U_1 \cap \cdots \cap U_k)\},$$

where $\sigma_y = (U_1, \ldots, U_k)$ is the carrier of y. This is a closed subspace of $N\mathcal{U} \times X$. The second coordinate projection $\omega_{\mathcal{E}}\colon L_{\mathcal{U}}(X) \to X$ is a homotopy equivalence because X is normal. The homotopy inverse is $\psi_{\mathcal{U}} \colon X \to L_{\mathcal{U}}(X)$, $x \mapsto (g_{\mathcal{U}}(x), x)$, where $g_{\mathcal{U}} \colon X \to N\mathcal{U}$ is the canonical map associated to a partition of unity. Let $\pi^{\mathcal{V}}_{\mathcal{U}} \colon L_{\mathcal{V}}(X) \to L_{\mathcal{U}}(X)$ be the restriction of the map $\pi^{\mathcal{V}}_{\mathcal{U}} \times \mathrm{id} \colon N_{\mathcal{V}}(X) \times X \to N_{\mathcal{U}}(X) \times X$. Then

$$N(X) = \varprojlim_{\mathcal{U} \in \mathrm{Cov}^{\mathrm{s}} X} \{L_{\mathcal{U}}(X), \pi^{\mathcal{V}}_{\mathcal{U}}\},$$

$\omega_{\mathcal{U}} \circ \pi^{\mathcal{V}}_{\mathcal{U}} = \omega_{\mathcal{V}}$, and the map $N(X) \to X$ induced by $\{\omega_{\mathcal{U}}\}$ coincides with ω. Since each $\omega_{\mathcal{U}}$ is a homotopy equivalence,

$$\omega_* \colon \check{h}\left(\varprojlim_{\mathcal{U} \in \mathrm{Cov}^{\mathrm{s}} X} L_{\mathcal{U}}(X); K^{\varpi} R \right) \longrightarrow \check{h}(X; K^{\varpi} R)$$

is an isomorphism (e.g., once again using Lemma 7.4.3). Passing to another (homotopy) colimit, one gets the result for $K(R)$ instead of the semi-connective $K^{\varpi}(R)$. This verifies Axiom C.

7.5. REMARKS ABOUT TOPOLOGICAL PROPERTIES

Remark 7.5.1. $\widehat{X}$ is nonmetrizable for the same reasons as $\widehat{\mathbb{E}} - \mathbb{E}$ in Example 6.1.12 with the lexicographic order topology: both are compact but not separable. Note also that the action of Γ at infinity is large, and although $\widehat{X}$ happens to be Čech-acyclic, it is unlikely to be contractible. These three features of $\widehat{X}$ make Theorem 1 inapplicable.

Remark 7.5.2. This is related to the previous remark. Observe that in the case $G = \mathrm{SL}_2$ (discussed in detail in Example 6.1.12) the identification map $\widehat{X} \to \bar{X}^+$ can be factored through another compactification of $\bar{X}$ where all irrational strata are collapsed to points. All of our arguments can be done for that space. The matters can be simplified even further by noticing that the action of Γ on the resulting space is small at infinity, and the space $\widehat{X}$ itself is metrizable. Note, however, that this cannot be arranged in our more general situation because the action of Γ_{B_0} on $\nu(B_0)$ is already not small.

8. Bounded Saturation in the Boundary

8.1. THE METRIC IN $\bar{X}$

The space $\widehat{X}$ contains $\bar{X}$ as an open dense Γ-subset, in particular Γ acts continuously on $\bar{X}$ as before. The metric that we use in $\bar{X}$ is a transported Γ-invariant metric. It can be obtained by first introducing any bounded metric in the compact $\bar{X}/\Gamma$ and then taking the metric in $\bar{X}$ to be the induced path metric where the measured path-lengths are the lengths of the images in $\bar{X}/\Gamma$ under the covering projection.

In this situation, the diameter of a chosen fundamental domain Δ is bounded by some number D as is also the diameter of any Γ-translate of the domain. Notice that this metric is very different from the one Borel and Serre used in [8, Sect. 8.3]. The general metrization theorems of Palais they used produce metrics which are bounded at infinity.

The crucial property of our metric is that by choosing a base point x_0 in Δ and taking its orbit under the Γ-action we can embed the group Γ with the word metric quasi-isometrically in $\bar{X}$. In this sense, the metric is similar to the left invariant metric in a nilpotent Lie group used in Section 5.2.

8.2. FUNDAMENTAL DOMAINS AND SETS

If X is the symmetric space G/K for a linear semi-simple Lie group G, $\pi\colon G \to G/K$ is the natural projection, and $\Gamma < G$ is a discrete subgroup, G and Γ act on X from the left. Reembed Γ in G by conjugating by an element of K so that $\pi(e) \neq \gamma \cdot \pi(e)$ for any $\gamma \in \Gamma$, $\gamma \neq e$. Recall that X has a left G-invariant metric ds^2, and there is the corresponding distance function $d\colon X^2 \to \mathbb{R}_{\geqslant 0}$. Define

$$\mathcal{E} = \{x \in X : d(\pi(e), \gamma \cdot x) \geq d(\pi(e), x),\ \gamma \in \Gamma\}.$$

This set is called the *Poincaré fundamental domain*.

DEFINITION 8.2.1. Let G be a reductive $\mathbb{Q}$-group, and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. Then $\Omega \subset G(\mathbb{R})$ is a *fundamental set* for Γ if (1) $K \cdot \Omega = \Omega$ for a suitable maximal compact subgroup $K \subset G(\mathbb{R})$, (2) $\Gamma \cdot \Omega = G(\mathbb{R})$, (3) $\Omega^{-1}\Omega \cap (xG(\mathbb{Z})y)$ is finite for all x, y in $G(\mathbb{Q})$.

Remark 8.2.2. Property (1) implies that the image of Ω in $X = G(\mathbb{R})/K$ is a fundamental set for the induced action of Γ on X. If Ω is a fundamental set for $\Gamma = G(\mathbb{Z})$ then the property (3) allows to construct a fundamental set for any subgroup Γ' commensurable with Γ by taking $\Omega' = \bigcup_{\sigma \in \Sigma} \sigma \cdot \Omega$, where Σ is a set of representatives of $\Gamma'/(\Gamma \cap \Gamma')$. The classical constructions of Siegel fundamental sets can be seen in [46, Sects 4.2, 4.3].

Let P_0 be the standard minimal parabolic $\mathbb{Q}$-subgroup of G, let A be the maximal $\mathbb{Q}$-split torus of G contained in P_0, and K be the maximal compact subgroup in $G(\mathbb{R})$ whose Lie algebra is orthogonal (relative to the Killing form) to the Lie algebra of $A(\mathbb{R})$. Let

$$A_t = \{a \in A(\mathbb{R})^0 : \alpha(a) \leqslant t, \forall \alpha \in \Delta\}.$$

Recall that $P_0 = Z_G(A) \cdot R_u(P_0)$. Furthermore, $Z_G(A) \approx A \cdot F$ where F is the largest connected $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of $Z_G(A)$. From the Iwasawa decomposition, $G(\mathbb{R}) = K \cdot P(\mathbb{R})$. This yields the following decomposition: $G(\mathbb{R}) = K \cdot A(\mathbb{R})^0 \cdot F(\mathbb{R}) \cdot R_u P_0(\mathbb{R})$.

DEFINITION 8.2.3. A *Siegel set* in $G(\mathbb{R})$ is a set of the form

$$\Sigma_{t,\eta,\omega} = K \cdot A_t \cdot \eta \cdot \omega,$$

where η and ω are compact subsets of $F(\mathbb{R})$ and $R_u P_0(\mathbb{R})$ respectively.

THEOREM 8.2.4 (Borel). *There are a Siegel set $\Sigma = \Sigma_{t,\eta,\omega}$ and a finite set $C \subset G(\mathbb{Q})$ such that $\Omega = C \cdot \Sigma$ is a fundamental set for Γ.*

The results of Garland and Raghunathan from [30] determine the form of the Poincaré fundamental domains which look like the sets described in Theorem 8.2.4. They work with nonuniform lattices Γ in a rank one linear algebraic semi-simple Lie group G.

Fix some Iwasawa decomposition $G = KA_0N_0$. There is a parabolic subgroup $P_0 < G$ with Langlands decomposition $P_0 = M_0A_0N_0$.

THEOREM 8.2.5 (Selberg, Garland–Raghunathan).

1. *The total number of geodesic rays $r(t)$, $t \in \mathbb{R}_{\geqslant 0}$, such that $r(0) = \pi(e)$, $r(\mathbb{R}_{\geqslant 0}) \subset \mathcal{E}$, is finite. Denote the minimal such number by M and choose $\mathcal{E}$ with this number of cusps.*
2. *If $r_i = \lim_{t\to\infty} r_i(t) \in \partial X$, $1 \leqslant i \leqslant M$, let $\Gamma_i = \mathrm{stab}(r_i)$. Then $\mathcal{E} = \bigcup_{i=0}^{M} \mathcal{E}_i$, where $\mathcal{E}_0$ is a compact set, and there is $t_0 > 0$ such that*
$$r_i(\mathbb{R}_{\geqslant t_0}) \subset \mathcal{E}_i,\ \mathcal{E}_i \cap \mathcal{E}_j = \emptyset \text{ for } 0 \neq i \neq j \neq 0,\ \mathcal{E}_i = g_i\Sigma_i,$$
where $g_i \in G$, $g_i \notin \Gamma\backslash e$, and $\Sigma_i = \{x \in X : x = r(\mathbb{R}_{\geqslant l})$ for geodesic rays $r\colon y \to W(P_0)$, $y \in \omega_i$, $\omega_i \subset X$ are compact$\}$.
3. *One has $g_i\Gamma_i g_i^{-1} < M_0N_0$, where N_0 is the maximal nilpotent subgroup of the stabilizer of the standard cusp, and ω_i is the closure of a fundamental domain for $g_i\Gamma_i g_i^{-1}$ in the horocycle $N_0 \cdot r(l)$.*

When Γ is an arithmetic subgroup, the cusps r_i are rational, i.e., $g_i \in G(\mathbb{Q})$. If Γ is torsion-free then $g_i\Gamma_i g_i^{-1}$ acts freely in $N \cdot r(l)$. Also, there is $\omega = \omega_i$, $1 \leqslant i \leqslant M$. Consider $\Gamma' = \langle \Gamma, g_1, \ldots, g_M \rangle < G(\mathbb{Q})$, the subgroup generated by the listed elements. This subgroup has a fundamental domain with unique cusp which is contained in a Siegel set $\Sigma_{t,\omega}$, ω being the closure of a fundamental domain of $\Gamma' \cap N_0$ in $N_0(\mathbb{R}) \cong e(P_0)$. So $\Delta \subset q_{P_0}^{-1}(\omega)$ and $(\Gamma'/\Gamma' \cap N_0) \cdot \mathrm{cl}(\Delta) = q_{P_0}^{-1}(\omega)$. According to part (3) of Theorem 8.2.5, Δ can be completed to the fundamental domain $\bar{\Delta}$ of Γ in $\bar{X}$ so that $\mathrm{cl}_{\bar{X}}(\bar{\Delta}) = \mathrm{cl}_X(\Delta) \cup \omega$.

8.3. QUASI-ISOMETRY INVARIANCE

Every two arithmetic subgroups Γ_1, Γ_2 in G are commensurable, hence their Cayley graphs are quasi-isometric. This also implies that if d_i are Γ_i-invariant metrics in $\bar{X}$ transported from $\bar{X}/\Gamma_i$, $i = 1, 2$, then $(\bar{X}, d_1)$ and $(\bar{X}, d_2)$ are quasi-isometric.

PROPOSITION 8.3.1. *The system of boundedly saturated sets in $\widehat{X} - \bar{X}$ is a quasi-isometry invariant of $(\bar{X}, d)$.*

Proof. Let $(\bar{X}, d_1)$, $(\bar{X}, d_2)$ be quasi-isometric structures on $\bar{X}$. It suffices to show that for a subset $\Omega \subset \bar{X}$ and a large $D_1 \gg 0$, the enlargement $\Omega[D_1]_1$ with respect to d_1 is contained in $\Omega[D_2]_2$ for some $D_2 > 0$. If λ, ϵ are the constants associated to the quasi-isometry id: $(\bar{X}, d_2) \to (\bar{X}, d_1)$, let $D_2 = (D_1 - \epsilon)/\lambda$. Then $x \in \Omega[D_2]_2 \Rightarrow d_2(x, o) \leqslant D_2$, $o \in \Omega \Rightarrow d_1(x, 0) \leqslant \lambda d_2(x, o) + \epsilon = D_1$.

Recall $\Gamma' < G(\mathbb{Q})$ constructed in Section 8.2. Since $g_i \Sigma_{l,\omega} g_i^{-1}$ are precisely the parabolic vertices of $\mathcal{E}$, the complement $\mathcal{E} \backslash \bigcup_{i=1}^{M} g_i \Delta g_i^{-1}$ is compact, so a Γ-domain is contained in $\bar{\Delta}[D]_{\Gamma'}$ for some $D > 0$. This implies even more directly that the boundedly saturated sets determined by Γ and Γ' coincide. Now we can study the bounded saturation using the simpler domain $\bar{\Delta}$.

8.4. SATURATION IN RATIONAL STRATA

Fix the coordinate map $\varrho \overset{\text{def}}{=} \sigma^{-1}: \mathbb{R}^r \to N_0 \cong e(P_0)$ defined in Section 4.2. Let $\mathcal{O} = \{(x_i) \in \mathbb{R}^r : 0 \leqslant x_i \leqslant 1, \forall 1 \leqslant i \leqslant r\}$, then $\varrho(\mathcal{O})$ is a domain for $\Gamma' \cap N_0$ in $e(P_0)$. The translates form a cellular decomposition of $e(P_0)$. The induced decompositions of $e(P)$, $P = gP_0 g^{-1} \in \mathcal{P}_{\mathbb{Q}}$, are invariant under $\Gamma' \cap N$, hence are well-defined.

Let $Z_i \overset{\text{def}}{=} \langle \gamma_i \rangle = G_i \cap \Gamma'_P$, where $\Gamma'_P = \Gamma' \cap P(\mathbb{Q})$. The computation in Section 4.2 shows that the union of translates of the fundamental cube $g\varrho(\mathcal{O})g^{-1}$ in $e(P)$ by the coset Γ'_P/Z_i disconnects $e(P)$. If $\chi_i: \mathbb{Z} \to Z_i$ is the obvious isomorphism then $\chi_i(n) \cdot \Gamma'_P/Z_i$ also disconnect $e(P)$. We will call these unions of cells *walls* in $e(P)$ and denote them by $\mathcal{W}_{i,n}$.

PROPOSITION 8.4.1. *Each cell in the decomposition of $\tau(P)$ from Section 4.2 for $P \in \mathcal{P}_{\mathbb{Q}}$ is boundedly saturated in Y.*

Proof. The closures of walls in $e(P)$ disconnect $\nu(P)$. The complements of $\mathcal{W}_{i,n}$ are denoted by $\mathcal{R}^{\pm}_{i,n}$. Note that the cell in $\tau(P)$ corresponding to the i-th coordinate and the positive or negative direction is the inverse limit of $\mathcal{R}^{\pm}_{i,n}$, $n \in \mathbb{Z}$, ordered by inclusion. Choose a cell σ by fixing i and $+$, loosing no generality. If $y \in \delta X \backslash \nu(P)$, say $y \in \nu(P')$, then the geodesic asymptotic to both $W(P)$ and $W(P')$ projects to $\bar{y} \in e(P)$. Then $\bar{y} \in \mathcal{W}_{i,n}$ for some $n \in \mathbb{Z}$. Denote $\mathcal{C}(\mathrm{int}\,(\mathcal{W}_{i,n+1} \cup \mathcal{W}_{i,n+2} \cup \mathcal{W}_{i,n+3}))$ by $\mathcal{B}_{i,n+2}$. If the subset $\Xi \subset \Gamma' \cap N_P$ makes $\mathcal{W}_{i,n+2} = \Xi \cdot \omega$ then $(\Gamma'/\Gamma' \cap N)\, \Xi \cdot \bar{\Delta} \subset \mathcal{B}_{i,n+2}$, so $\mathcal{B}_{i,n+2}$ is a *barrier* separating y and σ into the different connected components of $\delta X \backslash \mathcal{B}_{n,m+2}$: $y \in \mathcal{H}^{-}_{i,n+2}$, $\sigma \subset \mathcal{H}^{+}_{i,n+2}$. If $\{y_s\} \subset \mathcal{H}^{-}_{i,n+2}$ is a sequence converging to y then $\overline{\{y_s\}[1]} \cap \mathcal{H}^{+}_{i,n+2} = \emptyset$. Inductively $\overline{\{y_s\}[D]} \cap \mathcal{H}^{+}_{i,n+3D} = \emptyset$, therefore, $\overline{\{y_s\}[D]} \cap \sigma = \emptyset$. By Lemma 5.1.2, σ is boundedly saturated.

8.5. INTERSTRATOUS SATURATION

Two sequences $\{x_i^1\}$, $\{x_i^2\}$ in a metric space (X, d) are called *fellow travelers* if there is $K > 0$ such that $d(x_i^1, x_i^2) \leqslant K$ for every $i \in \mathbb{N}$.

LEMMA 8.5.1. *Let $\{y_i\}$ and $\{z_i\}$ be sequences in $(\bar{X}, d)$ converging to $y \in \nu(P_y) \cap Y$, $z \in \nu(P_z) \cap Y$. If $P_y \neq P_z$ then the sequences do not fellow-travel.*

Proof. Let $\bar{y}_i$, $\bar{z}_i$ be the points in the image of the imbedding $\iota\colon \Gamma' \hookrightarrow X$, $\gamma \mapsto \gamma \cdot x_0$, $x_0 \in \Delta \subset \bar{\Delta}$, in the same translate of the domain $\bar{\Delta}$ as y_i and z_i. Since $d(y_i, \bar{y}_i) \leqslant D$, $d(z_i, \bar{z}_i) \leqslant D$, it suffices to show that $\{\bar{y}_i\}$, $\{\bar{z}_i\}$ do not fellow-travel.

Suppose that $\bar{y}_i = \iota(\gamma_i')$, $\bar{z}_i = \iota(\gamma_i')$. Observe that if the sequences $\{\bar{y}_i\}$, $\{\bar{z}_i\}$ fellow-travel in the Γ-invariant metric d_Γ then they also fellow-travel in the Riemannian metric d_G. Indeed, in the Γ-invariant metric $d(\bar{y}_i, \bar{z}_i) = d((\gamma_i')^{-1}(\bar{y}_i), (\gamma_i')^{-1}(\bar{z}_i)) = d(\imath(I), (\gamma_i')^{-1}(\bar{z}_i))$. Since there is a constant M such that $d(\bar{y}_i, \bar{z}_i) \leqslant M$, all of $(\gamma_i')^{-1}(\bar{z}_i)$ are contained in a word-metric ball in $\imath(\Gamma)$ of radius M centered at $\imath(I)$. They form a finite set which is, therefore, bounded in the Riemannian metric d_G in X which is G-invariant. So there is a constant N such that $d_G(\bar{y}_i, \bar{z}_i) = d_G(\imath(I), (\gamma_i')^{-1}(\bar{z}_i)) \leqslant N$.

Now each translate $\gamma \cdot \bar{\Delta}$ contains at most a finite number of points $\{y_i\}$ for otherwise $y \in \gamma \cdot \omega \subset \gamma \cdot \bar{\Delta} \subset \mathsf{C}Y$. Thus the sequence $\{\bar{y}_i\}$ takes on infinitely many values. By inspection of projections into $e(P_y)$, $\lim_{i \to \infty}\{\bar{y}_i\} = W(P_y) \in \partial X$. Same argument shows that the limit of $\{\bar{z}_i\}$ in εX is $W(P_z)$. This shows that the sequences $\{\bar{y}_i\}$, $\{\bar{z}_i\}$ do not fellow-travel in the Riemannian metric; neither do they in our metric d_Γ by the observation above.

COROLLARY 8.5.2. *Each stratum-component $\varsigma(P) = \nu(P) \cap Y$, $P \in \mathcal{P}_\mathbb{R}$, of Y is boundedly saturated.*

Proof. If $\varsigma(P)$ is not boundedly saturated then there are fellow-traveling sequences $\{y_i\}$, $\{z_i\}$ converging to $y \in \varsigma(P)$ and $z \notin \varsigma(P)$ (see the proof of Lemma 5.1.2). This is impossible by Lemma 8.5.1.

That the boundaries of rational strata are boundedly saturated is not new: this follows from Proposition 8.4.1 and Corollary 5.1.4. But Corollary 8.5.2 also says that each $\nu(P)$, $P = \mathcal{P}_\mathbb{R} \backslash \mathcal{P}_\mathbb{Q}$, is boundedly saturated.

Remark 8.5.3. It is impossible to use the theorems of Section 5 about right actions here: the right action of $G(\mathbb{R})$ on X does not extend to $\bar{X}$. For example, in the SL_2-situation, in the upper-half plane model, the image of the y-axis

$$iy \cdot \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = y \left(\frac{\beta\delta + \alpha\gamma + i}{\delta^2 + \gamma^2} \right)$$

is a straight Euclidean line with slope $1/(\beta\delta + \alpha\gamma)$, not a geodesic. It is interesting to note, however, that there are constructions of Mumford et al. ([3], [39, IV, Sect. 2]) to which the right action naturally extends. Compare the pictures on page

179 in [39]. The lines in the second picture, if extended, should converge at the other endpoint of the horizontal geodesic. They are precisely the images of the right action computed above.

9. Proof of Theorem 3

9.1. ORDERLY COVERINGS

Given a covering $\mathcal{U} \in \mathrm{Cov}^{\mathrm{s}}\,\widehat{X}$, the proof of compactness of $\widehat{X}$ (Section 7.3) required a construction of a finite covering $\{V(P_i)\}$ of ∂X associated to some regular neighborhoods $\mathrm{Reg}_{n_i}(P_i)$ of $\tau(P_i)$. We want a variation on that constuction which, given a covering $\mathcal{U} \in \mathrm{Cov}^{\mathrm{s}}\,Y$, refines $\mathcal{U}$. Recall from Section 5.5 that $\mathrm{Star}^{\mathrm{o}}_n(v)$ is the open star of $v \in \Upsilon\rho(V_{(n)}) \subset \nu(P)$, $P \in \mathcal{P}_{\mathbb{R}}$, in the projection of the nth derived cubical decomposition of I^r.

DEFINITION 9.1.1. Let us alter the notation $\mathcal{C}(\mathrm{Star}^{\mathrm{o}}_n(v))$ in this definition to mean $\mathrm{Star}^{\mathrm{o}}_n(v)$ when $v \in e(P)$. Define

$$\mathrm{Ord}_n(v) = Y \cap \mathcal{C}(\mathrm{Star}^{\mathrm{o}}_n(v)) \setminus \bigcup_{P' \in \mathcal{F}} \nu(P'),$$

where $\mathcal{F}$ is the set of all P' with $q_P(W(P')) \in \partial\, \mathrm{Star}^{\mathrm{o}}_n(v)$.

It is clear that $\mathrm{Ord}_n(v)$ is an open neighborhood of v in Y. For any covering $\mathcal{U} \in \mathrm{Cov}^{\mathrm{s}}\,Y$ there is an order n such that $\{\mathrm{Ord}_n(v) : v \in \Upsilon\rho(V_{(n)}) \cap Y\}$ refines $\{U \in \mathcal{U} : U \cap \nu(P) \neq \emptyset\}$. Now it is clear from compactness of Y that there is a finite set $\{P_k : k \in \Lambda\} \subset \mathcal{P}_{\mathbb{R}}$ and integers n_k so that $\bigcup_k \{\mathrm{Ord}_{n_k}(v) : v \in \Upsilon\rho(V_{(n_k)}) \subset \nu(P_k)\}$ refines the given $\mathcal{U} \in \mathrm{Cov}^{\mathrm{s}}\,Y$. The full cofinal subcategory of $\mathrm{Cov}^{\mathrm{s}}\,Y$ consisting of such *orderly* refinements will be denoted by $Ord^{\mathrm{s}}Y$.

In order to create manageable rigid coverings, we consider the excised versions of the sets $\mathrm{Ord}_n(v)$:

$$\mathrm{ExcOrd}_n(v) = \mathrm{Ord}_n(v) \backslash \mathrm{Star}^{\mathrm{o}}_n(v).$$

Now $ExcOrd^{\mathrm{s}}Y$ is the category of open coverings $\mathcal{V}$ which contain some $\mathcal{U} \in Ord^{\mathrm{s}}Y$ as a subset and may contain $\mathrm{ExcOrd}_n(v)$ if $\mathrm{Ord}_n(v) \in \mathcal{U}$. The cofinality property mentioned above is certainly not affected.

DEFINITION 9.1.2. Let $\mathrm{P{\small RE}O{\small RD}}Y$ be the full subcategory of $Cov\,Y$ with objects $\beta \in \mathrm{P{\small RE}O{\small RD}}\,Y$ satisfying

– $\mathrm{im}\,\beta \in ExcOrd^{\mathrm{s}}\,Y$,
– $y \in \nu(P_k)$ for some $k \Longleftrightarrow \beta(y) = \mathrm{Ord}_{n_k}(v)$ for some $v \in \nu(P_k)$.

It is implicit in the second condition that for $y \notin \nu(P_k)$ for all k, there exists $\ell \in \Lambda$ and n_ℓ with $\beta(y) = \mathrm{ExcOrd}_{n_\ell}(v)$ for some $v \in \nu(P_\ell)$. Define $Ord\,Y$ to be the full subcategory of $Cov\,Y$ closed under $\times$-operation generated by $\mathrm{P{\small RE}O{\small RD}}\,Y$.

It is easy to see that $Ord\, Y$ is not cofinal in $Cov\, Y$ but satisfies the hypotheses on the category $\mathcal{C}$ in Section 2.3. Recall that the conclusion of that section was that the map

$$\jmath^*\colon \check{h}(Y;KR) \longrightarrow \underset{\overleftarrow{Ord\, Y}}{\mathrm{holim}}(N\_ \wedge KR)$$

induced by the inclusion $\jmath\colon Ord\, Y \hookrightarrow Cov\, Y$ is a weak homotopy equivalence.

9.2. DEFINITION OF $\{\alpha\}$

The idea is to define finite rigid coverings of Y by boundedly saturated sets which can be naturally 'piecewise' approximated by coverings from $Ord\, Y$. Let $N_0 = N_{P_0}$ for the standard $P_0 \in \mathcal{P}_{\mathbb{Q}}$. Consider the covering of ∂I^r by the 2^r open stars of $V_{(-1)}$ in the (-1)-st derived decomposition. The images $\Upsilon_0\rho(\mathrm{Star}^{\mathrm{o}}(v_{-1}(s_1,\ldots,s_r)))$ cover the boundary τN_0. The sets are no longer open but they are boundedly saturated with respect to the Γ-invariant metric as a consequence of Proposition 8.4.1 and Corollary 5.1.4.

DEFINITION 9.2.1. The covering $\mathcal{A}_0$ of τN_0 by the sets

$$\Upsilon_0\rho(\mathrm{Star}^{\mathrm{o}}(v_{-1}(s_1,\ldots,s_r)))$$

is finite but not open. This choice generates the category $\{\alpha_0\}$ of finite rigid coverings α_0 of τN_0. Notice that it follows from property (2) of finite rigid coverings that $\mathrm{im}\,\alpha_0 = \mathcal{A}_0$.

Notice that the homotopy type of $N\mathcal{A}_0$ is, in fact, that of the $(r-1)$-dimensional sphere: the nerve of $\mathcal{A}_0$ is the same as the nerve of the open star covering of ∂I^r with respect to the (-1)-st derived decomposition, and that can be easily seen to be homotopy equivalent to S^{r-1}.

The choice of $\mathcal{A}_0$ provides well-defined coverings $\mathcal{A}_P$ of $\nu(P)$, $P \in \mathcal{P}_{\mathbb{Q}}$, by $G(\mathbb{Z})$-translates of $\mathcal{A}_0$. There are also associated rigid coverings $\{\alpha_P\}$ with $\mathrm{im}\,\alpha_P = \mathcal{A}_P$.

DEFINITION 9.2.2. Given a covering $\omega \in Ord\, Y$, $\omega = \pi_1 \times \ldots \times \pi_m$, where each $\pi_i \in$ PREORDY. Let $\{P_k : k \in \Lambda\}$ be the finite collection of parabolic $\mathbb{R}$-subgroups associated to $\pi_1,\ldots,\pi_m$. Collect the following data:

1. for each $P \in \mathcal{P}_{\mathbb{Q}}$ pick an arbitrary $\alpha_P \in \{\alpha_P\}$ – in particular, for each $P_k \in \mathcal{P}_{\mathbb{Q}}$ there is $\alpha_k \in \{\alpha_{P_k}\}$,
2. for each $P \notin \mathcal{P}_{\mathbb{Q}}$ take α_P to be the constant rigid covering with $\mathrm{im}\,\alpha_P = \nu(P)$.

Define the following finite rigid covering $\alpha(\omega,\alpha_P)$:

$$\alpha(y) = \begin{cases} \alpha_k(y) \cup (\omega(y) \backslash \nu(P_k)) & \text{if } y \in \varepsilon(P_k) \text{ for some } k \in \Lambda, \\ \omega(y) & \text{otherwise.} \end{cases}$$

Since each $\omega(y)$, $y \notin \nu(P_k)$ for $k \in \Lambda$, is a union of closed strata $\nu(P) \cap Y$, it is boundedly saturated by Corollary 8.5.2. Same is true about $\omega(y) \backslash \nu(P_k)$, $y \in \nu(P_k)$. Also, all $\operatorname{im} \alpha_k$, $k \in \Lambda$, are boundedly saturated by Proposition 8.4.1 and Corollary 5.1.4. Thus all $\alpha(y)$, $y \in Y$, are boundedly saturated subsets of Y.

9.3. PROOF OF THEOREM 3

Now we can return to the argument started in Section 3.4 and use $\mathcal{C} = Ord\, Y$ and $\alpha' = \alpha(\omega, \alpha_P)$ for $\omega \in Ord\, Y$. We have seen that $\jmath^*$ is a weak equivalence. It remains to see that the orderly sets are nice enough for all inclusions $\mathsf{sat}_*\colon N\beta \hookrightarrow N\alpha(\beta)$ to be homotopy equivalences.

This is the way one would proceed if there were no need to make the construction of $\widehat{X}$ equivariant. Instead of $\nu(B_0)$ we would use nonequivariant but simpler compactifications by cubes of appropriate dimensions. With the obvious choices of cubical derived decompositions (induced by Υ which is now a cellular homeomorphism) and the other constructions repeated literally, the saturation process in the boundaries of rational strata would produce sets which are stars of lower-dimensional sides. It would be enough to consider the stars of the vertices.

Now notice that there is a projection of this hypothetical situation to the real equivariant Y. This projection induces an equivalence on the Čech homology level by our weak Vietoris–Begle theorem 7.4.1. Also, the images of the saturations in the hypothetical boundary Y^{hyp} project to precisely the boundedly saturated sets we construct in Y. There is a well-defined functorial 'lift' from our α's to the saturations in Y^{hyp} with the same combinatorics. The induced maps form a commutative diagram:

$$\begin{array}{ccc}
\underset{\overleftarrow{Ord\, Y^{\mathrm{hyp}}}}{\operatorname{holim}}\, (N_ \wedge KR) & \xrightarrow{\mathsf{sat}_*} & \underset{\overleftarrow{Ord\, Y^{\mathrm{hyp}}}}{\operatorname{holim}}\, N\alpha(_) \wedge KR \\
\Big\downarrow \simeq & & \Big\downarrow \simeq \\
\underset{\overleftarrow{Ord\, Y}}{\operatorname{holim}} (N_ \wedge KR) & \xrightarrow{\alpha_*} & \underset{\overleftarrow{Ord\, Y}}{\operatorname{holim}} N\alpha(_) \wedge KR
\end{array}$$

So our constructions induce precisely the needed map. Now inclusions of nerves $N\beta \hookrightarrow N\alpha(\beta)$ induce natural weak equivalences $N\beta \wedge KR \simeq N\alpha(\beta) \wedge KR$. This follows from the fact that factoring out a contractible subcomplex generated by a subset of vertices factors through the inclusion into the complex where the same subset generates a simplex. This is precisely what happens with finitely many disjoint subcomplexes associated to sets covering the special strata. We can conclude that α_* is a weak homotopy equivalence by Theorem 1.1.2.

Remark. It is easy to see that the obvious reconstruction of ω from $\alpha(\omega, \alpha_P)$ which 'forgets' about the choices of α_k in $\{\alpha_{P_k}\}$ defines a functor $R\colon \{\alpha\} \to \mathit{Ord}\, Y$. This is the exact inverse to α. The natural induced map

$$R^*\colon \operatorname*{holim}_{\overleftarrow{\mathit{Ord}\, Y}} (N\alpha\_ \wedge KR) \longrightarrow \operatorname*{holim}_{\overleftarrow{\{\alpha\}}} (N\alpha R\_ \wedge KR) = \operatorname*{holim}_{\overleftarrow{\{\alpha\}}} (N\_ \wedge KR)$$

is a weak equivalence of spectra by Theorem 1.1.3. Now the composition

$$R^* \alpha_* \jmath^*\colon \check{h}(Y; KR) \xrightarrow{\simeq} \operatorname*{holim}_{\overleftarrow{\alpha \in \{\alpha\}}} (N\alpha \wedge KR)$$

is a weak homotopy equivalence with the required target.

Appendix A. Other Theories. Other Groups. Other Methods

A.1. EXTENSIONS TO OTHER THEORIES

The extension of the K-theoretic results to L-theory is formal using the basic results of [16, Sects 4, 5]. The statements about the L-theoretic assembly maps are the same as before when the coefficient spectrum is replaced by the nonconnective spectrum $L^{-\infty}(R)$ for a ring with involution R satisfying $K_{-i}(R) = 0$ for sufficiently large i. The homotopy groups $\pi_i(L^{-\infty}(R))$ are the surgery obstruction groups $L_i(R)$.

The extension to A-theory is trickier. The necessary details are provided by [19] and earlier papers of W. Vogell.

If $C^*(\Gamma)$ denotes the group C^*-algebra of Γ (the completion of $L^1(\Gamma)$ in the greatest C^*-norm), Kasparov defines $\alpha\colon RK_*(B\Gamma) \longrightarrow K_*(C^*(\Gamma))$. The splitting of this map implies the Novikov conjecture for Γ—see the explanation on page 414 of [49] or Corollary 2.10 in [50]. The recent work of Carlsson–Pedersen–Roe [18] extends the methods used here to work for this C^*-algebraic version of α.

A.2 HILBERT MODULAR GROUPS

Let F be a totally real algebraic number field of degree n over $\mathbb{Q}$, let $\mathcal{O}_F$ be the ring of integers of F. Consider $G = R_{F/\mathbb{Q}}\mathrm{SL}_2$, the $\mathbb{Q}$-group obtained from SL_2/F by restriction of scalars according to Weil ([56, Section 1.3]). Then $G(\mathbb{Q}) = \mathrm{SL}_2(F)$, $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})^n$ is a connected semi-simple Lie group, $K = \mathrm{SO}(2)^n$ is a maximal compact subgroup, and the associated symmetric space $X(G) = \mathbb{H}^n$ has rank n. Any subgroup of finite index in $G(\mathbb{Z}) = \mathrm{SL}_2(\mathcal{O}_F)$ is an irreducible lattice in $G(\mathbb{R})$ embedded via the inclusion $\mathrm{SL}_2(\mathcal{O}_F) \hookrightarrow \mathrm{SL}_2(\mathbb{R})^n$ by using the n distinct $\mathbb{Q}$-homomorphisms $F \to \mathbb{R}$ as coordinate functions.

The *Hilbert modular groups* are $\mathrm{SL}_2(\mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers in the real quadratic field $\mathbb{Q}(\sqrt{d})$. Here the two homomorphisms $\mathcal{O}_d \to \mathbb{R}$ are the inclusion and the Galois conjugation. We will assume that Γ is a neat arithmetic subgroup of $\mathrm{SL}_2(\mathcal{O}_d)$. (A subgroup Γ is *neat* when the subgroup of $\mathbb{C}^*$ generated by the eigenvalue of any element of Γ is torsion-free. In particular, Γ itself is torsion-free.) The quotients $\Gamma\backslash X$ are called the Hilbert(–Blumenthal) modular surfaces.

Examples of such arithmetic subgroups of G are the *principal congruence subgroups*

$$\Gamma[\ell] = \ker\Big(\mathrm{SL}_2(\mathcal{O}_d) \xrightarrow{\text{mod}} \mathrm{SL}_2(\mathcal{O}_d/(\ell))\Big)$$

for $\ell \geqslant 3$.

THEOREM A.2.1. *If Γ is a neat arithmetic subgroup of $R_{\mathcal{O}_d/\mathbb{Q}}\mathrm{SL}_2$ then the assembly map α is a split injection.*

If P_0 is the standard parabolic $\mathbb{Q}$-subgroup of SL_2 then $B_0 = P_0 \times P_0$ is the standard parabolic $\mathbb{Q}$-subgroup of $G = R_{\mathcal{O}_d/\mathbb{Q}}\mathrm{SL}_2$. The $\mathbb{Q}$-rank of G is one. The stabilizer of the standard cusp in Γ is a uniform lattice in the solvable Lie group Sol, and the associated stratum in the Borel–Serre enlargement can be identified with the underlying space of Sol ([26], [34, Sect. 3.H]) where the stabilizer acts by left multiplication.

The group Sol can be expressed as a semi-direct product of $\mathbb{R}^2$ and $\mathbb{R}$: if the elements of the set Sol $= \mathbb{R}^2 \times \mathbb{R}$ are (x, y, z), the action of z is the linear transformation given by $(x, y) \mapsto (\mathrm{e}^z \cdot x, \mathrm{e}^{-z} \cdot y)$. We can transport the flat metric from $\mathbb{R}^3$ into Sol using this identification. The straight lines through the origin in Sol are then given as

$$L = \{(x_1^b + tx_1^d, x_2^b + tx_2^d, x_3^b + tx_3^d) : t \in \mathbb{R}\}.$$

Here is the formula for the left action of $(y_1, y_2, y_3) \in$ Sol on this line:

$$(y_1, y_2, y_3) \circ L = (y_1 + \mathrm{e}^{y_3}(x_1^b + tx_1^d), y_2 + \mathrm{e}^{-y_3}(x_2^b + tx_2^d), y_3 + x_3^b + tx_3^d).$$

It shows that Sol acts on the parallelism classes of rays in the stratum. The right multiplication action of Sol on itself does not extend to the parallelism classes of rays. There is one set of lines, however, invariant under the right action: if $x_3^d = 0$ then

$$L \circ (y_1, y_2, y_3) = (x_1^b + tx_1^d + \mathrm{e}^{x_3^b}y_1, x_2^b + tx_2^d + \mathrm{e}^{-x_3^b}y_2, x_3^b + y_3).$$

The formula also shows that each class of lines in this set is actually fixed by the right action.

Now consider the ideal compactification of Sol with the flat metric. Each point in $\partial(\mathrm{Sol})$ with $x_3^d = 0$ can be blown up to a closed segment, the interior points corresponding to subclasses of lines with the common coordinate $-\infty < x_3^b < +\infty$. The result will be called $\nu(\mathrm{Sol})$.

The same methods as in Section 5 apply and show that each open segment is boundedly saturated as well as each of the endpoints and each of the complementary hemispheres. The closed segments above are the elements of a cylinder

$\mathrm{Sol} \cup [-\infty, +\infty] \times \partial D^2 \subset \nu(\mathrm{Sol})$. Let us identify $\nu(\mathrm{Sol})$ with the closed cylinder $[-\infty, +\infty] \times D^2$ and embed $\nu(\mathrm{Sol})$ in $\mathbb{R}^3$ with cylindrical coordinates (r, θ, z) as the set defined by $0 \leqslant r \leqslant 1$, $0 \leqslant \theta \leqslant 2\pi$, and $0 \leqslant z \leqslant 1$. Given a natural number n, define the *n-th standard sectoral decomposition* of $\nu(\mathrm{Sol})$ to be the representation of $\nu(\mathrm{Sol})$ as the union of sectors $\mathcal{S}_n(i, j, k) = \{(r, \theta, z) \in \mathbb{R}^3 : i/2^n \leqslant r \leqslant (i + 1)/2^n,\ j\pi/2^{n-1} \leqslant \theta \leq (j + 1)\pi/2^{n-1},\ k/2^n \leqslant z \leqslant (k + 1)/2^n\}$ for every choice of the integral triple $0 \leqslant i, j, k \leqslant 2^n - 1$. The points $v_n(i, j, k) = (i/2^n,\ j\pi/2^{n-1},\ k/2^n)$ for $0 \leqslant i, j, k \leqslant 2^n$ will be called *vertices*. A vertex in the N-th subdivision determines star $\mathrm{Star}(v_n(i, j, k)) = \{(r, \theta, z) \in \mathbb{R}^3 : (i - 1)/2^n \leqslant r \leqslant (i+1)/2^n,\ (j-1)\pi/2^{n-1} \leqslant \theta \leq (j+1)\pi/2^{n-1},\ (k-1)/2^n \leqslant z \leqslant (k + 1)/2^n\}$ with the obvious modifications when i or k equals 0 or 1. Also *links* and *open stars* are defined by direct analogy with their simplicial analogues.

The boundary set δX is the union of the rational strata

$$\delta_{\mathbb{Q}} X \stackrel{\mathrm{def}}{=} \nu(\mathrm{Sol}) \times_{B_0(\mathbb{Q})} G(\mathbb{Q})$$

and the irrational points at infinity with the auxiliary topology defined by the obvious analogy with Section **??**. Now $\widehat{X}_{\mathrm{b}}$ is the analogue of $\widehat{X}$ or, more precisely, $\widehat{X}_{\mathrm{b}}$ from Remark 7.2.1. The basic neighborhoods of irrational points at infinity are completions of their neighborhoods in the spherical topology.

Using the argument from Remark 7.2.1, it is easy to see that $\widehat{X}_{\mathrm{b}}$ is compact but not Hausdorff due to the arrangement of higher dimensional maximal flats in X (see Remark 7.2.1). In order to induce the Hausdorff property, consider the set map $f \colon \widehat{X}_{\mathrm{b}} \to \varepsilon X$. The idea is to make this map continuous. Introduce a new topology in $\widehat{X}_{\mathrm{b}}$ generated by the intersections of basic neighborhoods $\mathcal{N}(x)$, $x \in \widehat{X}_{\mathrm{b}}$, and the preimages of neighborhoods of $f(x) \in \varepsilon X$. Since each fiber of f is Hausdorff, and the analogue of Lemma 7.1.3 holds, the new topology on $\widehat{X}_{\mathrm{b}}$ is Hausdorff and makes f a quotient map. Denote the new space by $\widehat{X}$. The map $f \colon \widehat{X} \to \varepsilon X$ can be used as in Section 7.4 to show that $\widehat{X}$ is Čech-acyclic.

The rest of the argument for the $\mathbb{R}$-rank one case generalizes easily, we only need to indicate the boundedly saturated sets we choose inside the rational boundary strata. It suffices to show the subsets of $\widehat{e}(B_0) \cong \nu(\mathrm{Sol})$. For the chosen $n \in 2\mathbb{N}$ and $\xi \in \{0, 1\}$, they are $A(n, j, \xi) = \{(r, \theta, \xi)\} \cup \{(r, \theta, z) : r = 1,\ (2j + \xi - 1)\pi/2^{n-1} \leqslant \theta \leq (2j + \xi + 1)\pi/2^{n-1},\ z \in (1 - \xi, \xi]\}$, where $0 \leqslant j \leqslant (\sqrt{2})^n - 1$ is an integer. These are open stars of certain collections of vertices in the $2n$-th standard sectoral decomposition.

Remark A.2.2.

1. The construction of the map f is apparently the correct way to deal with the general case of a lattice in a Hermitian symmetric domain. The target must be the maximal Satake compactification which coincides with εX in the rank one situation (cf. [32, Sect. 10.3, Sect. 12.5, Appendix D]).

2. The ad hoc construction of $\nu(\mathrm{Sol})$ is designed to be analogous to the NIL case. The correct way to deal with Sol $\cong e(B_0)$ is, of course, using [33, Lemma (7.8)] mentioned before.

A.3 OTHER APPROACHES TO NOVIKOV CONJECTURES

FerWei There has been a lot of research done on Novikov and related conjectures. The most recent progress known to us is connected with the work of Bökstedt–Hsiang–Madsen, Carlsson–Pedersen, Connes–Gromov–Moscovici, Farrell–Jones, Ferry–Weinberger, Higson–Roe, Julg–Kasparov, Ogle, and others.

The method of S. Ferry and S. Weinberger ([27, 28, 29]) uses a similar 'bounded control philosophy'. They call an endomorphism of a metric space $f\colon X \to X$ *bounded* if there is $k > 0$ such that $d(f(x), x) < k$ for all $x \in X$.

THEOREM A.3.1 (Ferry–Weinberger). *If Γ is a discrete group such that $K = K(\Gamma, 1)$ is a finite complex and the universal cover $X = \tilde{K}$ has a compactification $\widehat{X}$ with the properties that*

1. *the boundary $\widehat{X} - X \subset \widehat{X}$ is a Z-set, i.e., admits a homotopy $F_t\colon \widehat{X} \to \widehat{X}$ with $F_0 = \mathrm{id}$, $F_t(\widehat{X}) \subset X$ for all $t > 0$, and*
2. *every continuous bounded function $f\colon X \to X$ extends by identity to a continuous function $\widehat{f}\colon \widehat{X} \to \widehat{X}$,*

then the L- and A-theoretic Novikov conjectures for Γ hold.

We wish to describe one difficulty in using our compactification in this approach. The Lie algebra n of the nilpotent radical of $P \in \mathcal{P}_{\mathbb{Q}}$ decomposes into a direct sum $\mathsf{n} = \mathsf{n}_\lambda \oplus \mathsf{n}_{2\lambda}$, where λ is the *unique* simple root of (P, A) – a consequence of the $\mathbb{R}$-rank one assumption. The dimensions $\dim(\mathsf{n}_\lambda)$, $\dim(\mathsf{n}_{2\lambda})$ equal the multiplicities of λ, 2λ. After exponentiating we get $N = N_\lambda N_{2\lambda}$ with $N_\lambda \cap N_{2\lambda} = \{I\}$ and $N_{2\lambda} = [N, N]$. If $N_{2\lambda} \neq \{I\}$ then N is a non-Abelian two-step nilpotent group with center $N_{2\lambda}$.

In the situation when N is non-Abelian, there exists an element $g \in N$ such that g acts nontrivially from the right on τN. This action $\tau\psi_g\colon \tau N \to \tau N$ is the extension from the action ψ_g on N which in its turn extends to a bounded endomorphism Ψ of $\bar{X}$. The point is that Ψ cannot be extended to an endomorphism of $\widehat{X}$ by identity on Y, so even our compactification of $e(P)$ cannot be used here. However, this obstacle disappears in the case when N is Abelian, for instance, in the case of $G = \mathrm{SO}_0(n, 1)$.

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